PHYS 5210

Graduate Classical Mechanics

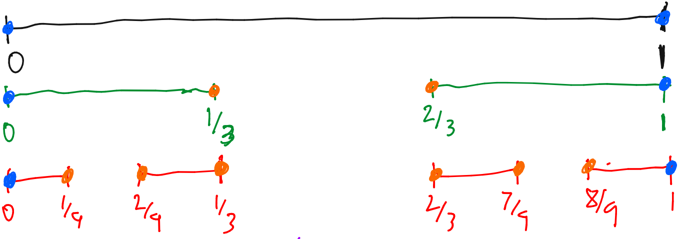
Fall 2022

Lecture 39

Fractals

November 28

1 What is the Cantor set? What is its volume?



$S_0 = [0,1]$

$S_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

S_2

$\vdots$

Cantor set: $S = \lim_{n\to\infty} S_n$

volume $V(S_0) = \int_{S_0} dx = \int_0^1 dx = 1$

$V(S_1) = 2/3$

$\vdots$

$V(S_n) = \frac{2}{3} V(S_{n-1}) = \left(\frac{2}{3}\right)^n$

$\vdots$ $\underline{V(S) \le V(S_n)}$.

$V(S) = 0$.

• S exists, not empty.

$0 \in S$

$1 \in S$

$1/3 \in S$

$2/3 \in S$

since only middle third of intervals deleted...

S has ∞ # of points

uncountable

$\boxed{2}$ Fractals are sets/surfaces/etc. with complex/intricate structure at all scales.

e.g. Cantor set

what is length of coast?

L_{coast} isn't well defined?

- answer depends on ruler length...

$L_{ruler} \to 0$, $L_{coast} \to \infty$.

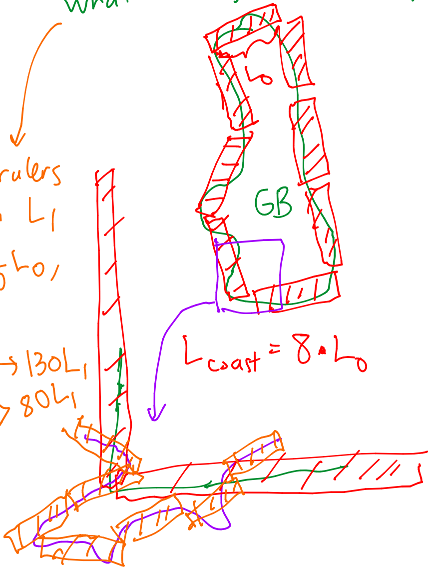
now w/ rulers of length L_1

$L_1 = \frac{1}{10} L_0$,

find

$L_{coast} \to 130 L_1$

$> 80 L_1$

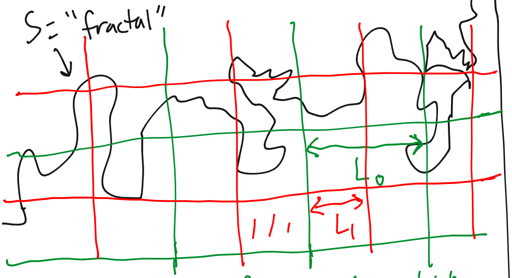


Fractal characterized by non-integer dimension.

many defs. for dimension

this class: box dimension

3 The box dimension allows us to characterize the dimension of a fractal.



N_0 = # of boxes of size L_0 which contain a point in S.

$(= 8)$

N_1 = " " ... of size L_1

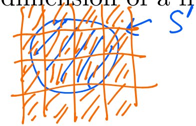
$< 32.$

If N_n = # of boxes filled ... at size L_n

Define box dimension:

$$d = \lim_{L_n \to 0} \frac{\log N_n}{\log 1/L_n}$$

Ex 1:



as $L_n \to 0$

$$N_n \to \pi \left(\frac{R}{L_n}\right)^2 + \ldots$$

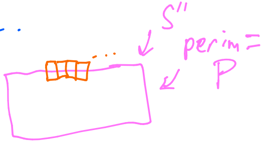
$$d(S') = \lim \frac{\log\left(\frac{\pi R^2}{L_n^2} + \cancel{\ldots}\right)}{\log 1/L_n}$$

$\log \frac{1}{L_n^2}$

$$= \lim \frac{\log(\pi R^2) + 2\log\frac{1}{L_n}}{\log 1/L_n}$$

$$= 2.$$

Ex 2:



$$N_n \approx \frac{P}{L_n}$$

$\to d = 1$

4 Calculate the box dimension of the Cantor set.

S_2

What's the natural box size L_n? $L_n = 3^{-n}$.

Each time we lay down rulers of length L_n, only "non-deleted thirds" contribute...

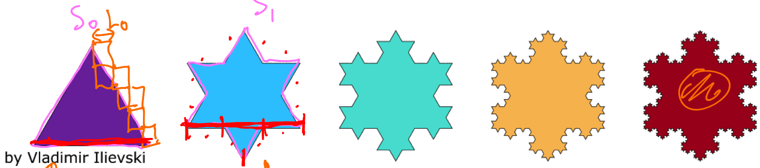
$$\underbrace{N_n = \frac{1}{L_n} V(S_n)}_{V(S_n) = L_n \cdot N_n} = 2^n.$$

log = ln

$$d = \lim_{L_n \to 0} \frac{\log N_n}{\log 1/L_n} = \lim_{n\to\infty} \frac{\log 2^n}{\log 3^n} = \frac{\cancel{n}\log 2}{\cancel{n}\log 3}$$

$$\approx 0.63.$$

5 Calculate the box dimension of the Koch snowflake.



$L_1 = L_0/3$

$N_0 \approx \underbrace{\sim 10}_{\text{boxes per segment}} \cdot 3 = 30$

$$N_1 = \sim 10 \cdot N_{\text{segments in } S_1}$$

$$\sim 10 \cdot 12 \sim 120$$

$$\underline{= 4N_0}$$

Iterate again: $N_n \approx 4^n N_0$

$$d = \lim_{n\to\infty} \frac{\log(4^n N_0)}{\log\left(\frac{1}{3^n}\right)^{-1}}$$

$$= \frac{\log 4}{\log 3} \approx 1.26$$