PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 4

Relativistic charged particles; Translation symmetry

August 29

1 Review special relativity.

Translation symmetry: $x^\mu \to x^\mu + a^\mu$

$$x^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Lorentz transform: $x^\mu \to \Lambda^\mu{}_\nu x^\nu$ $\quad \sum_{\nu = t,x,y,z}$

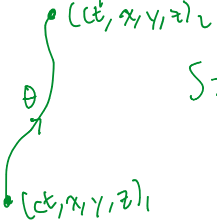
(boost & rotation)

where $\Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \eta^{\mu\nu} = \eta^{\rho\sigma}$

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Action of a free particle:

arbitrary parameterization θ:



$$S = -mc \int d\theta \sqrt{-\frac{dx_\mu}{d\theta}\frac{dx^\mu}{d\theta}}$$

$\theta = t$

$$S = -mc^2 \int dt \sqrt{1 - \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{c^2}}$$

$$= -mc^2 \int dt \sqrt{1 - \frac{\dot{x}_i \dot{x}^i}{c^2}}$$

i index: only x, y, z.

2 Review the vector potential of electromagnetism.

Physically, measure $\vec{E}$ & $\vec{B}$ (E_i) (B_i). But some constraints:

$$\left.\begin{array}{l} \nabla\cdot\vec{B}=0 \\ \nabla\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t}\end{array}\right\}$$

Introduce: $\vec{A}$ and Φ :

$$\vec{B}=\nabla\times\vec{A} \qquad \left[\varepsilon_{ijk}\,\partial_j A_k = B_i\right.$$

$$\vec{E}=-\frac{\partial\vec{A}}{\partial t}-\nabla\Phi \qquad \left.E_i=-\frac{\partial A_i}{\partial t}-\partial_i\Phi\right]$$

Still some redundancy: $\vec{A}=\vec{A}+\nabla\lambda$, and $\Phi\to\Phi-\dfrac{\partial\lambda}{\partial t}$

$\vec{B}\to\vec{B}$ and $\vec{E}\to\vec{E}$. (gauge transform)

$$A_\mu=\begin{pmatrix}-\Phi/c\\ A_x\\ A_y\\ A_z\end{pmatrix} \qquad A_\mu\to\Lambda_\mu{}^\nu A_\nu$$

Electromagnetic Fields:

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$$

$$=\begin{pmatrix}0 & -E_x/c & -E_y/c & -E_z/c\\ E_x/c & 0 & B_z & -B_y\\ E_y/c & -B_z & 0 & B_x\\ E_z/c & B_y & -B_x & 0\end{pmatrix}$$

Gauge transform:

$$A_\mu\to A_\mu+\partial_\mu\lambda$$

$$\left[\partial_\mu=\begin{pmatrix}\frac{1}{c}\frac{\partial}{\partial t}\\ \partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{pmatrix}\right]$$

3 Find the Lagrangian. Use gauge invariance to explain why other subleading corrections in A are not allowed.

$\int dx^\mu A_\mu$

Claim: gauge invariance of E&M fixes

$$S = -mc\int d\theta \sqrt{-\frac{dx_\mu}{d\theta}\frac{dx^\mu}{d\theta}} + q\int d\theta \frac{dx^\mu}{d\theta} A_\mu(x(\theta))$$

q ↑ charge of particle

Why? $A_\mu \to A_\mu + \partial_\mu \lambda$:

$$S \to S + \int d\theta \underbrace{\frac{dx^\mu}{d\theta}\frac{\partial}{\partial x^\mu}\lambda(x(\theta))}$$

$$\frac{d\lambda(x^\mu)}{d\theta} = \frac{dx^\mu}{d\theta}\frac{\partial\lambda}{\partial x^\mu}$$

$$\int d\theta \frac{d}{d\theta}\lambda = \lambda(\theta_f) - \lambda(\theta_i)$$

Why not $\int d\theta \left[\frac{dx^\mu}{d\theta} A_\mu\right]\left[F_{\mu\nu}F^{\mu\nu}\right]$

$$S \to S + \int d\theta \frac{d\lambda}{d\theta}(F^2) \neq \lambda F^2\Big|_{\theta_i}^{\theta_f}$$

'voltage'

$\theta = t$:
$$S = \int dt\left[-mc^2\sqrt{1-\frac{\dot{x}_i\dot{x}^i}{c^2}} - q\Phi + q\dot{x}^i A_i\right]$$

$$\dot{x}_i\dot{x}^i = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

4 Find the equations of motion for a relativistic charged particle.

$$S = \int dt \left[-mc^2 \sqrt{1 - \frac{\dot{x}_j \dot{x}^j}{c^2}} - q\Phi(x) + q\dot{x}^j A_j(x) \right]$$

identity

$$\left(\frac{\partial \dot{x}^j}{\partial \dot{x}_i} = \delta^{ji} \right)$$

EOMs: $\frac{\delta S}{\delta x_i(t)} = \frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0$

$\frac{\partial}{\partial x_i}$

$$\frac{\partial L}{\partial x_i} = -q\partial_i \Phi + q\dot{x}^j \partial_i A_j$$

$$\frac{\partial L}{\partial \dot{x}_i} = mc^2 \cdot \frac{2\dot{x}_i/c^2}{2\sqrt{1 - \frac{\dot{x}_i \dot{x}^i}{c^2}}} + qA_i$$

$$p_i = \frac{m\dot{x}_i}{\sqrt{1 - \dot{x}^2/c^2}}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = \frac{dp_i}{dt} + q\frac{\partial A_i}{\partial t} + q\frac{\partial A_i}{\partial x_j}\frac{dx_j}{dt}$$

Combine: $0 = -q\partial_i \Phi + q\dot{x}^j \partial_i A_j - q\dot{x}^j \partial_j A_i - q\frac{\partial A_i}{\partial t} - \frac{dp_i}{dt}$

$$\frac{dp_i}{dt} = qE_i + q(\dot{x} \times B)_i$$

$q\,\varepsilon_{ijk}\dot{x}_j B_k$

Lorentz force!

Aside:

$$\frac{\delta S}{\delta x^\mu} = \frac{\partial L}{\partial x^\mu} + \cdots = \partial_\mu L + \cdots$$

$$\frac{dL}{d\theta} = \frac{dx^\mu}{d\theta}\frac{\partial L}{\partial x^\mu}$$

5 What is translation symmetry? What are its consequences for effective theory?

Suppose $L(x_i, \dot{x}_i)$ is Lagrangian

ONE coord

Suppose x_a has translation symmetry: $L(x_a + c) = L(x_a)$

rest

$x_i = (x_a, x_I)$

$$L(x_a + c, x_I, \dot{x}_a, \dot{x}_I) = L(x_a, x_I, \dot{x}_a, \dot{x}_I)$$

$$0 = \frac{L(x_a + c, \ldots) - L(x_a, \ldots)}{c} \text{ : take } c \to 0: \quad \frac{\partial L}{\partial x_a} = 0.$$

example of Noether's Thm: continous symmetry $\Updownarrow$ conservation law

Recall: EOM

$$\frac{\delta S}{\delta x_a} = 0 = \cancel{\frac{\partial L}{\partial x_a}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_a}$$

Define canonical momentum $\left(\frac{dp_a}{dt} = 0\right)$

$$p_a = \frac{\partial L}{\partial \dot{x}_a},$$

then trans. sym ensures p_a is conserved

Example: charged part x-electric field:

$$S = \int dt \left[-mc^2 \sqrt{1 - \frac{\dot{x}^2}{c^2}} + qEx\right]$$

inv. $y \to y + c$

$z \to z + c$

$$\frac{d}{dt} p_y = 0 \qquad p_y = \frac{m\dot{y}}{\sqrt{1 - \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{c^2}}}$$