PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 40

Fractal structure of attractors

November 30

1 Review the period doubling transition to chaos in the logistic map.

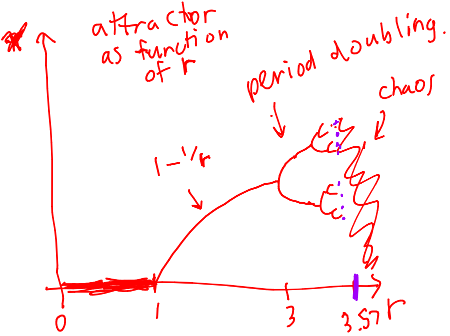
Logistic map: $x_{n+1} = r x_n (1-x_n)$ $\quad 0<r<4, \quad 0 \le x_n \le 1.$

(r: const.)

↳ dissipative map: as $n \to \infty$, generic x_0 will trend towards attractor $R \subset [0,1]$.

e.g. $1<r<3$: $R = \{1-\frac{1}{r}\}$ (stable fix pt)

$3<r<3.44\ldots$; R is 2 points (on stable period-2 cycle)

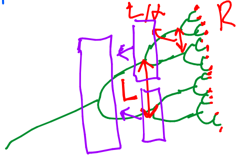


Claim: at $r = r_c \approx 3.57$, attractor of logistic map is fractal.

2 Estimate the box dimension of the attractor at r_c.

Box dimension: $d = \lim_{L \to 0} \frac{\log N}{\log 1/L}$ ← # of boxes of side length L overlapping R.

Math: (Hausdorff ≈ box) $d \approx 0.54$.



$\alpha \approx 2.5$

$N_n = 2^n$ ←

$\frac{L_n}{L_{n-1}} = \frac{1}{\alpha}$; $L_n = \frac{1}{\alpha^n}$

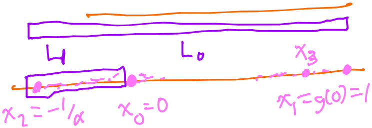
$$d = \lim_{n \to \infty} \frac{\log(2^n)}{\log(\alpha^n)} = \frac{\log 2}{\log \alpha} \approx 0.76$$

Better way (cf lec 38)

$x_{n+1} = g(x_n)$

$g(1) = -\frac{1}{\alpha} g(\alpha \cdot 0)$

$g(g(x)) = -\frac{1}{\alpha} g(\alpha x)$



$x_0, x_2, x_4 \dots$ ← rescaled version of $x_0, x_1, x_2, \dots$

too fast

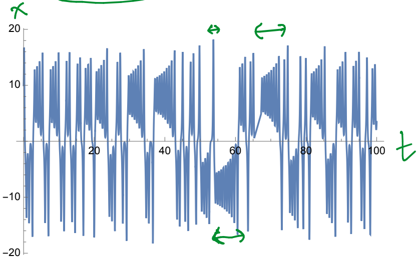
$$\frac{L_1}{L_0} = \frac{1/\alpha}{1 + 1/\alpha} = \frac{1}{\alpha + 1}$$

$$d = \frac{\log 2}{\log(\alpha+1)} \approx 0.55.$$

3 Describe the Lorenz equations. The dynamics at $\sigma = 10$, $r = 28$, $b = 2.67$ experiences intermittency.

dissipative ODEs.

Thm: ≥ 3 DOF needed for chaos.



Lorenz equations:

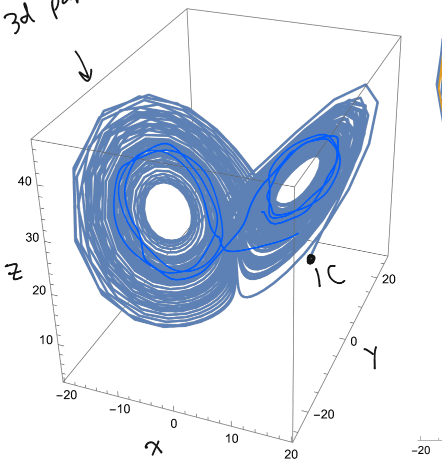
$$\dot{x} = \sigma(x - z)$$
$$\dot{y} = x(r - z) - y$$
$$\dot{z} = xy - bz$$

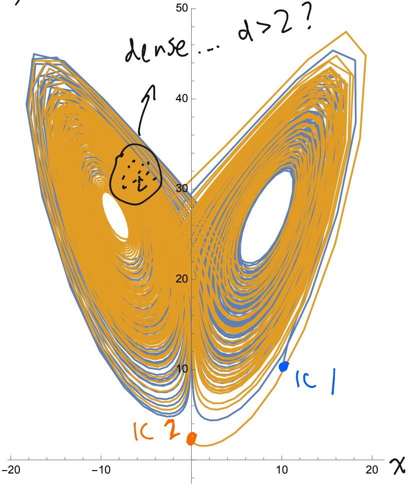
σ, r, b dimensionless #.

at late t, also approach strange attractor (fractal)

4 The dynamics at $\sigma = 10$, $r = 28$, $b = 2.67$, tends towards a strange attractor, independent of initial conditions.

3d parametric plot $(x(t), y(t), z(t))$





5 The fractal dimension of the Lorenz strange attractor is $d \approx 2.05$.