PHYS 5210

Graduate Classical Mechanics

Fall 2022

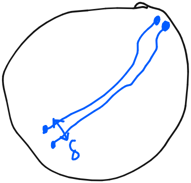
Lecture 41

The butterfly effect

December 2

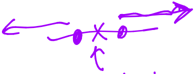
1 The butterfly effect is the sensitivity of chaotic dynamics to initial conditions.

consider generic initial conditions for map / ODE . . .



integrable (not chaos, at least)

NOTE: not describe dynamics near unstable fixed point.



unstable

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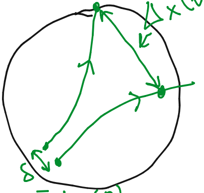
chaos

$\Delta x(t) \sim \underbrace{\delta e^{\lambda t}}_{\text{butterfly effect}}$



$$\frac{\partial \Delta x(t)}{\partial \Delta x(0)} = e^{\lambda t}$$

$\lambda =$ Lyapunov exponent.

2 The butterfly effect is visible in the chaotic region of the logistic map ($r = 3.6$)

$x_{n+1} = r x_n (1 - x_n)$, $\quad r = 3.6$

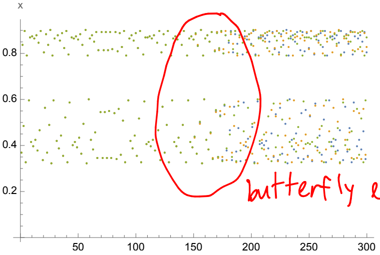
$0 \le x_{n+1} \le 1$

3 initial conditions

$x_0 = 0.37$

$x_0 = 0.37 + 10^{-15}$

$x_0 = 0.37 - 10^{-15}$



butterfly effect!

$n = t$

$n \sim 150$: $\Delta x_n \sim 1$ compared to $\Delta x_0 \sim 10^{-15}$

Estimate λ?

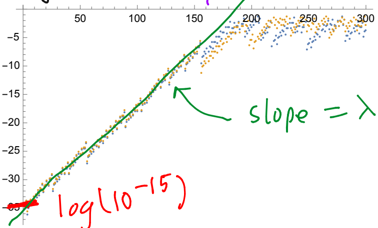
$$\frac{\overbrace{\Delta x_n}^{\sim 1}}{\Delta x_0 \sim 10^{-15}} = 10^{15} = e^{\overbrace{150}^{n}\lambda}$$

$$\lambda \approx \frac{15 \log 10}{150} \approx 0.23.$$

3 The Lyapunov exponent allows us to quantitatively measure the strength of the butterfly effect.

trajectories are $O(1)$ apart.

$\log(|\Delta x_n|)$



n

slope $= \lambda \simeq 0.23$

$\log(10^{-15})$

$$\Delta x_n = \Delta x_0 \cdot e^{\lambda n}$$

$$\log(\Delta x_n) = \log(\Delta x_0) + \lambda n.$$

$\boxed{4}$ $\lambda > 0$ heralds the onset of chaos.

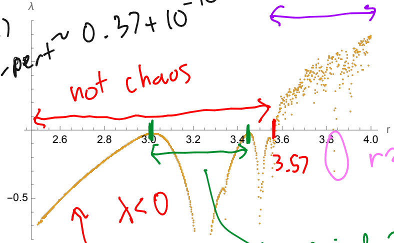
For many values of r:

$x_0^{(1)} \approx 0.37$ $\qquad x_{0,pert}^{(1)} \sim 0.37 + 10^{-10}$

$x_0^{(2)} \approx 0.8$

Calculate

$$\frac{1}{n} \log \left(\frac{x_n^{(i)} - x_{pert}^{(i)}}{(x_0^{(i)} - x_{0,pert}^{(i)})} \right) = \lambda$$



chaos $(\lambda > 0)$

not chaos

3.57

$\lambda < 0$

single stable fixed pt: λ = decay exp. towards stable fixed pt.

period-2 cycle that's stable.

Stable $r \approx 3.83$: period-3 cycle.