PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 42

The KAM Theorem

December 5

1 Resonances can lead to the breakdown of perturbation theory in action-angle coordinates.

Recall: integrable system has n conserved quantities $J_1, \dots J_n$

↳ action-angle coords (J_α, ϕ_α) : $H_0(J_\alpha)$:
$\alpha = 1 \dots n$

$$\dot{J}_\alpha = -\frac{\partial H}{\partial \phi_\alpha} = 0 \,, \quad \dot{\phi}_\alpha = \frac{\partial H}{\partial J_\alpha} = \omega_\alpha$$

const.

Perturb: $H = H_0 + \varepsilon H_1$, $H_1 = \sum_{m_1, m_2 \dots m_n} h_{\vec{m}} \, e^{i\vec{m}\cdot\vec{\phi}}$

Type 2 CT: $S_1 = \sum \frac{h_{\vec{m}}}{i\vec{m}\cdot\vec{\omega}} e^{i\vec{m}\cdot\vec{\phi}}$

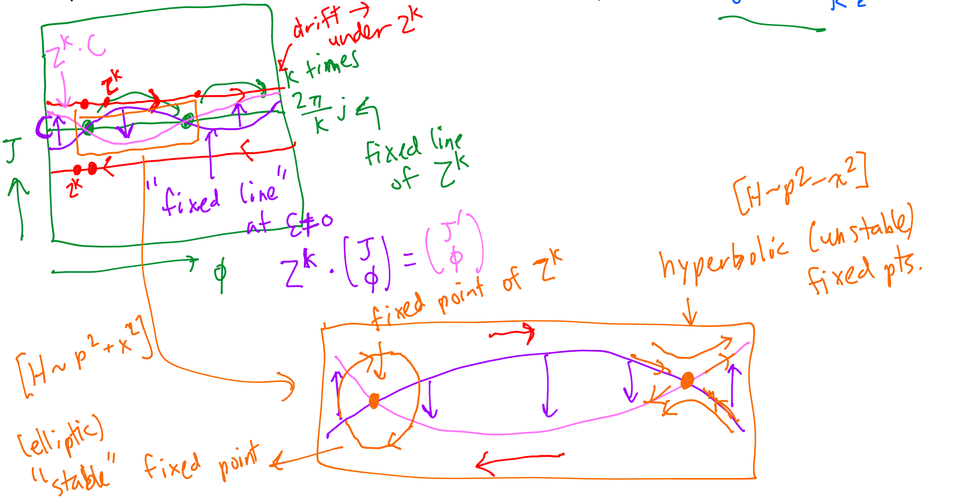
↳ if $\vec{m}\cdot\vec{\omega} = 0$: resonance (commensurate freq)

perturbation theory fails.

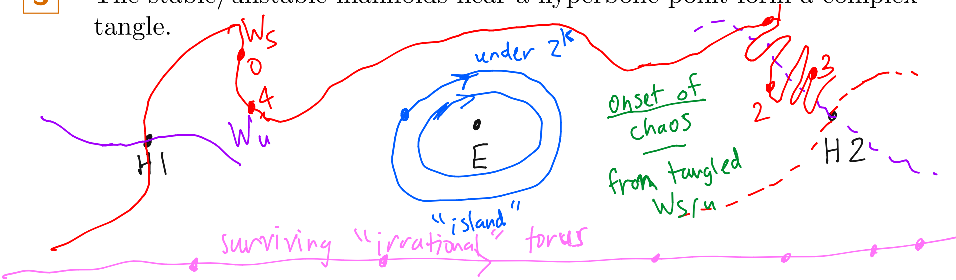
$\boxed{2}$ In a two-dimensional map, a rational torus will break up into elliptic and hyperbolic points under a perturbation.

$$J_{n+1} = J_n + \varepsilon \sin\phi_n \qquad = Z \cdot \begin{pmatrix} J_n \\ \phi_n \end{pmatrix}$$
$$\phi_{n+1} = \phi_n + J_{n+1}$$

If $\varepsilon = 0$, "integrable"
there are period-k cycles: $J_0 = 2\pi \frac{j}{k}$ integer.



3 The stable/unstable manifolds near a hyperbolic point form a complex tangle.



stable line ($W_{s,1}$ or $W_{s,2}$); points p where $(Z^k)^n \cdot p \to H1, 2$

unstable line ($W_{u,1}$ or $W_{u,2}$): points p where $(Z^k)^{-n} \cdot p \to H1, 2$

Math facts: • W_s / W_u can not self-intersect.

• $W_{s,1}$ and $W_{s,2}$ can not self-intersect.
(pts flow to either H1 or H2 [or neither])

• $W_{s,1}$ and $W_{u,2}$ can intersect.

• chaos at any $\varepsilon \neq 0$.

4 State the KAM Theorem.

KAM Thm: if $H_0 + \varepsilon H_1 = H$, for ε small enough, integrability in "dense" subset of phase space.

e.g. "more" irrational numbers than rational

K.R. $\downarrow$ not chaos $\qquad \downarrow$ chaos

#1: $\omega_\alpha = \frac{\partial H}{\partial J_\alpha}$ is invertible (locally)

$\det\left(\frac{\partial \omega_\alpha}{\partial J_\beta}\right) \neq 0.$

} mostly incommensurate frequencies.

#2: Recall: $S_1 \sim \sum e^{i\vec{m}\cdot\vec{\phi}} \frac{h_{\vec{m}}}{i\vec{m}\cdot\vec{\omega}} \to$ stays small.

$|\vec{m}\cdot\vec{\omega}| \geq C \cdot |\vec{m}|^{-K}$

ω's "sufficiently irrational"

5 Sketch how the Diophantine condition can hold.