PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 5

Noether's Theorem

August 31

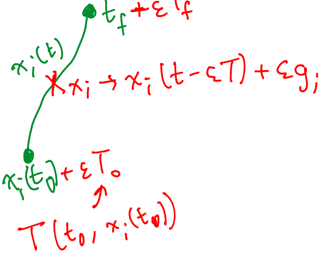
1 Noether's Theorem: every continuous symmetry has a corresponding conservation law.

$\varepsilon =$ infinitesimal

Action invariant under

$$\begin{cases} x_i \to x_i + \varepsilon\, g_i(t, x) \\ t \to t + \varepsilon\, T(t, x) \\ L \to L + \varepsilon\, d\Phi/dt \end{cases}$$

Assume $L(x_i, \dot{x}_i)$



$$S(\varepsilon) = \int_{t_i + \varepsilon T_i}^{t_f + \varepsilon T_f} dt \left[\underbrace{L(x_i, \dot{x}_i)}_{L\left(x_i(t-\varepsilon T),\ \frac{d}{dt} x_i(t-\varepsilon T)\right)} + \varepsilon \frac{d\Phi}{dt} \right]$$

Goal: Calculate $\dfrac{dS}{d\varepsilon} = 0$

Show: there exists conserved $Q(x_i, \dot{x}_i, t)$

$$\frac{d}{dt} Q = 0$$

2 Prove Noether's Theorem.

$$S(\varepsilon) = \int_{t_0+\varepsilon T_0}^{t_f+\varepsilon T_f} dt \left[L(x_i + \varepsilon g_i, \dot{x}_i + \varepsilon \dot{g}_i)\Big|_{t\to t-\varepsilon T} + \varepsilon \frac{d\Phi}{dt} \right]$$

$$L\left(x_i(t-\varepsilon T), \frac{d}{dt} x_i(t-\varepsilon T)\right)$$

$$= L + \frac{\partial L}{\partial x_i}(-T\dot{x}_i)\varepsilon + \frac{\partial L}{\partial \dot{x}_i}(-T\ddot{x}_i)\varepsilon - \varepsilon \dot{T} \dot{x}_i \frac{\partial L}{\partial \dot{x}_i}$$

$$0 = \frac{dS}{d\varepsilon}\Big|_{\varepsilon=0} = \underbrace{T_f L_f - T_0 L_0}_{TL|_0^f} + \int_{t_0}^{t_f} dt \left[g_i \frac{\partial L}{\partial x_i} + \dot{g}_i \frac{\partial L}{\partial \dot{x}_i} \right]$$

$$\left[g_i \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} + \dot{g}_i \frac{\partial L}{\partial \dot{x}_i} \right]$$

$$\int_{t_0}^{t_f} dt \, \frac{d}{dt}\left[g_i \frac{\partial L}{\partial \dot{x}_i} \right] = g_i \frac{\partial L}{\partial \dot{x}_i}\Big|_0^f$$

$$- \int_{t_0}^{t_f} dt \left[\dot{T} \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} + \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i}\right)\dot{x}_i T + \frac{\partial L}{\partial \dot{x}_i} \ddot{x}_i T \right]$$

$$= -T\dot{x}_i \frac{\partial L}{\partial \dot{x}_i}\Big|_0^f$$

$$0 = TL\Big|_0^f + g_i \frac{\partial L}{\partial \dot{x}_i}\Big|_0^f - T\dot{x}_i \frac{\partial L}{\partial \dot{x}_i}\Big|_0^f + \Phi\Big|_0^f$$

If $Q = T\left(L - \dot{x}_i \frac{\partial L}{\partial \dot{x}_i}\right) + g_i \frac{\partial L}{\partial \dot{x}_i} + \Phi$, then $\frac{dQ}{dt} = 0$ (on phys. traject.)

$\boxed{3}$ Discuss the consequences of translation and rotation symmetries.

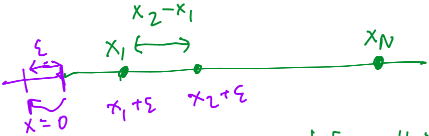
Translation (1 particle in 1d): $L(x, \dot{x})$ is invariant under $x \to x + \varepsilon$

def:

$(g = 1,\ T = 0,\ \Phi = 0)$: $Q = \dfrac{\partial L}{\partial \dot{x}}$ (momentum)

E.g. $L = -mc^2 \sqrt{1 - \dot{x}^2/c^2}$

$$\frac{\partial L}{\partial \dot{x}} = \frac{m\dot{x}}{\sqrt{1 - \dot{x}^2/c^2}}$$



Translation (many particles, 1 dim): $x_i \to x_i + \varepsilon$ (for all i)

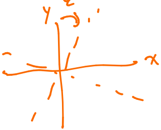
$(g_i = 1$ for all i, $T = 0,\ \Phi = 0)$: $\overset{P_{tot}}{\cancel{Q}} = \sum_{i=1}^{N} \dfrac{\partial L}{\partial \dot{x}_i}$

↳ angular mom. conservation.

Rotation (2d, single particle): $L\left(\sqrt{x^2+y^2}, \sqrt{\dot{x}^2+\dot{y}^2}\right)$

$x \to x + \varepsilon y$

$y \to y - \varepsilon x$:



$x^2 + y^2 \to (x + \varepsilon y)^2 + (y - \varepsilon x)^2 = x^2 + y^2 + \cancel{2x\varepsilon y} \cancel{- 2y\varepsilon x}$

$g_x = y$ $\qquad Q = y\, \partial L/\partial \dot{x} - x\, \partial L/\partial \dot{y}$

$g_y = -x$ $\qquad = y p_x - x p_y = -L_z$

4 Discuss the consequences of time-translation symmetry.

$$t \to t + \varepsilon \qquad [g_i = 0, \; T = 1, \; \Phi = 0]$$

$$Q = L - \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} = -E \quad \text{(energy)}$$

Example: rel. charged particle

$$\partial_t \Phi = \frac{\partial}{\partial t} A_i = 0$$

$$S = \int dt \left[-mc^2 \sqrt{1 - \frac{\dot{x}_i \dot{x}^i}{c^2}} - q\Phi + \underline{q\dot{x}^i A_i} \right]$$

$$E = \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} - L = \dot{x}_i p^i + mc^2 \sqrt{1 - \frac{\dot{x}_i \dot{x}^i}{c^2}} + q\Phi - \cancel{q\dot{x}^i A_i}$$

$$= \dot{x}_i \left[\frac{m\dot{x}^i}{\sqrt{1 - \dot{x}^2/c^2}} + \cancel{qA^i} \right] + \cdots$$

$$= q\Phi + \frac{mc^2}{\sqrt{1 - \dot{x}^2/c^2}} \left[\cancel{\frac{\dot{x}_i \dot{x}^i}{c^2}} + \left(1 - \cancel{\frac{\dot{x}_i \dot{x}^i}{c^2}}\right) \right]$$

electrostatic potential

$$= q\Phi + mc^2 / \sqrt{1 - \dot{x}_i \dot{x}^i / c^2}$$

$\leftarrow$ rest + kinetic

5 Discuss Galilean-invariant particle motion in one dimension.