PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 6

Boost symmetry; Configuration space

September 2

$\boxed{1}$ Review Noether's Theorem.

If action S invariant under
$$\begin{cases} x_i \to x_i + \varepsilon g_i \\ t \to t + \varepsilon T \\ L \to L + \varepsilon \frac{d\Phi}{dt} \end{cases}$$
ε = infinitesimal

then: $\frac{dQ}{dt} = 0$ on phys traj.

where $Q = T\left(L - \dot{x}_i \frac{\partial L}{\partial \dot{x}_i}\right) + g_i \frac{\partial L}{\partial \dot{x}_i} + \Phi$

Translation: $x \to x + \varepsilon$ $(g=1,\ T=0,\ \Phi=0)$: $Q = \frac{\partial L}{\partial \dot{x}} = p$ momentum

Rotation: $x \to x + \varepsilon y$, $y \to y - \varepsilon x$ $\quad \varepsilon_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \varepsilon$

or more general: $x_i \to x_i + \varepsilon_{ij} x_j$ $\quad (g_i = \varepsilon_{ij} x_j,\ T = \Phi = 0)$

$Q = p_i \varepsilon_{ij} x_j = L$ (angular momentum)

Time-trans: $t \to t + \varepsilon$ $\quad (g_i = 0,\ T = 1,\ \Phi = 0)$

$Q = -E = L - \dot{x}_i \frac{\partial L}{\partial \dot{x}_i}$ (energy)

$\boxed{2}$ Discuss Galilean-invariant particle motion in one dimension.

1) $x \to x + \varepsilon$ (trans)

2) $t \to t + \varepsilon$ (time trans)

3) $\underline{x \to x + \varepsilon t}$ (boost) AND $\Phi = -mx$

cf HW2; effective theory: $S = \int dt \, \frac{1}{2} m \dot{x}^2$

$$\dot{x} \to \dot{x} + \varepsilon$$

$$L \to \frac{1}{2} m (\dot{x} + \varepsilon)^2 \simeq L + \cancel{m\dot{x}}\varepsilon + \ldots + \varepsilon \frac{d\Phi}{dt} \to \cancel{-\varepsilon m \dot{x}}$$

$$S \to S + \int_{t_0}^{t_f} dt \, m \varepsilon \dot{x} = \cancel{S + m\varepsilon [x(t_f) - x(t_0)]}$$

Conserved quantity: $Q = t \frac{\partial L}{\partial \dot{x}} + \Phi = m(\dot{x} t - x)$

Why? Q is only conserved if $x(t) = a + bt$

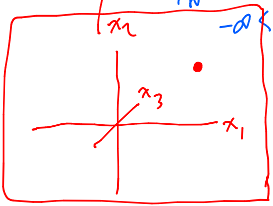
EoMs are $\ddot{x} = 0$

3 Define configuration space.

N particles on 1d line:

$$x_i = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \in \mathbb{R}^N$$

$-\infty < x_1, x_2, \ldots < \infty$



Goal: Lagr. mech. of systems where $x_i \notin \mathbb{R}^N$

Example:

Naively: $\theta \in \mathbb{R}$, $\theta \sim \theta + 2\pi$

identified w/

circle (S^1)

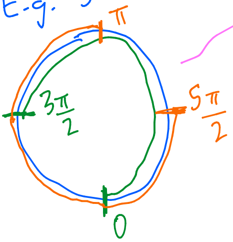
Configuration space: $M = \{$physically allowed "configuration"$\}$

M is a manifold

4 Why is configuration space a manifold?

Heuristically: manifold = space where "calculus exists"
(locally looks like $\mathbb{R}^N$)

E.g. S^1



$L(\theta, \dot{\theta}) = L(\theta + 2\pi, \dot{\theta})$

~~CAUTION: J/S says line interval is config. sp.~~

manifold = glue sets together, def. of der. agree.

Natural to extend POLA:

$$S : \{\text{trajectories on } M\} \to \mathbb{R}$$
(w/ Fixed BCs)

In practice, usually find coords, but global structure of M constrains Eff Th.