PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 7

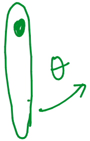
Constraints and Lagrange multipliers

September 7

1 Discuss dynamics on configuration space S^1.

Lagrangian mech makes sense on config. space

Example: S^1 (circle)



θ and $\theta + 2\pi$ are the same

Constraints on L?

-periodic: $L(\theta, \dot\theta) = L(\theta + 2\pi, \dot\theta)$

functions to include are $\sin(\theta)$, $\cos(\theta)$, $\dot\theta$ ← unconstrained!

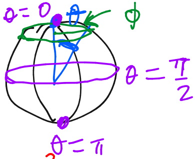
[products of these]

Example: $L = A\dot\theta^2 + B\cos\theta + \ldots$

$+ \dot\theta^3 + \ldots \dot\theta^2 \sin\theta + \ldots$

$\boxed{2}$ Discuss dynamics on configuration space S^2.

S^2 is the [2-dim] (surface of) sphere



Spherical coords
[restrict to $r=1$]

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right\}$$

What L makes sense?

$$L = \frac{m}{2}\left(\cancel{\dot{r}^2} + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) + \dots$$

$$\downarrow$$

$$L \sim \dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2$$

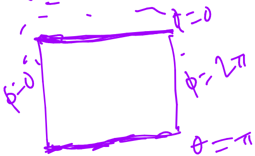
der. ind. terms

$$L(\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi)$$

Which points unique?

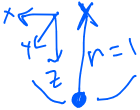
(θ, ϕ) eq. $(\theta, \phi + 2\pi)$

$0 \le \theta \le \pi$



3 Discuss the method of Lagrange multipliers.

(holonomic) constraints: $F(x_1, \dots, x_N) = 0$ must be obeyed



$$\underbrace{x^2 + y^2 + z^2 - 1 = 0.}$$

conf. space: $\mathbb{R}^3 \to S^2$

(x, y, z) $\qquad$ $x^2+y^2+z^2-1$

Lagrange multipliers: $L_{\mathbb{R}^3}(x_i, \dot{x}_i) + \underline{\lambda} F = L_{S^2}$

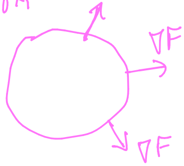
P.O.L.A: $0 = \underbrace{\frac{\delta S}{\delta x_i(t)}} = \frac{\delta S}{\delta \lambda(t)} \to \frac{\partial L}{\partial \lambda} - \cancel{\frac{d}{dt}\frac{\partial L}{\partial \dot{\lambda}}} = 0$

$\underline{F = 0}$

new: $\lambda \frac{\partial F}{\partial x_i}$

$$\frac{\partial}{\partial x_i}\left(L_{\mathbb{R}^3} + \lambda F\right) - \frac{d}{dt}\frac{\partial}{\partial \dot{x}_i}\left(L_{\mathbb{R}^3} + \cancel{\lambda F}\right) = 0$$

$\partial F / \partial x_i = 2x_i$



4 Use Lagrange multipliers to determine the action for a particle moving on S^2. $x_i x_i = 1$

$S^2 : \underbrace{x^2 + y^2 + z^2 - 1}_{} = 0$.

$L = L_{\mathbb{R}^3} + \lambda F, - V(x_i)$

$L_{\mathbb{R}^3} \to \frac{1}{2}\dot{x}_i \dot{x}_i$

Step 1: $\underbrace{x^2 + y^2} + z^2 = 1$

Use $\sin^2\theta + \cos^2\theta = 1$

$x^2 + y^2 = \sin^2\theta$

$\sin^2\theta\cos^2\phi \qquad \sin^2\theta\sin^2\phi$

1 constraint: $3 - 1 = 2$-dim remain

$[x = \sin\theta\cos\phi,\ y = \sin\theta\sin\phi,\ z = \cos\theta)$

$\dot{x}^2 + \dot{y}^2 + \dot{z}^2$

$= (\cos\theta\cos\phi\,\dot{\theta} - \sin\theta\sin\phi\,\dot{\phi})^2$

$+$ 2 more terms

$= \dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2$

Note: as before, $\sin\theta\,\dot{\phi}$ always shows up

$V(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

5 How can we remove the Lagrange multiplier from the equations of motion?

$$L = \frac{1}{2} m \dot{x}_i \dot{x}_i + \lambda (x_i x_i - 1)$$

??

$$\frac{\delta S}{\delta x_i} = \frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 2\lambda x_i - m \ddot{x}_i = \vec{0}$$

How to remove λ?

Use $x_i x_i = 1$

$$x_i \left[2\lambda x_i - m\ddot{x}_i \right] = 2\lambda - m \ddot{x}_i x_i = 0$$

Solve for $\lambda = \frac{1}{2} m \ddot{x}_i x_i$

$$m\ddot{x}_i = x_i \underbrace{(m \ddot{x}_j x_j)}_{\text{normal/constraint force}}$$