PHYS 5210

Graduate Classical Mechanics

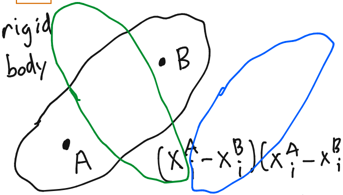
Fall 2022

Lecture 8

Configuration space of rigid body rotation

September 9

1 Why is the configuration space of rigid body motion $\mathbb{R}^3 \times \mathrm{SO}(3)$?



$(x^A_i - x^B_i)(x^A_i - x^B_i) = \text{const.}$

$(x^A - x^B)^2 + (y^A - y^B)^2 + (z^A - z^B)^2 = \text{const.}$

Translation: $\vec{x}_A \to \vec{x}_A + \vec{c}$

$\vec{x}_B \to \vec{x}_B + \vec{c}$

3x3 matrix

Rotation: $\vec{x}_A \to R\,\vec{x}_A$

$\vec{x}_B \to R\,\vec{x}_B$

$$\left[\begin{array}{l} x^A_i \to R_{ij}\, x^A_j \\ x^B_i \to R_{ij}\, x^B_j \end{array}\right]$$

$$(x^A_i - x^B_i)(x^A_i - x^B_i) = [R_{ij}(x^A_j - x^B_j)][R_{ik}(x^A_k - x^B_k)]$$

$$= \delta_{jk}(x^A_j - x^B_j)(x^A_k - x^B_k)$$

$R_{ij}R_{ik} = \delta_{jk}$ orthogonal

$$(\vec{x}^A - \vec{x}^B)^T(\vec{x}_A - \vec{x}_B) = [R(\vec{x}^A - \vec{x}^B)]^T R(\vec{x}^A - \vec{x}^B)$$

$$= (\vec{x}^A - \vec{x}^B)^T R^T R(\vec{x}^A - \vec{x}^B)$$

$R^T R = 1$ (identity)

Summary: config space

$M = \{(\vec{c}, R)\}$

$\mathbb{R}^3 \times \mathrm{SO}(3)$

2 Discuss the properties of SO(3).

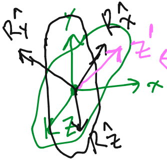
orthogonal matrices: $R_{ik}R_{ij} = \delta_{jk}$

two "types":

$R^T R = 1$

$\det(R)^2 = \det(R^T)\det(R) = \det(1) = 1$

$\det(R) = \pm 1.$



right-handed coords:

$\hat{x} \times \hat{y} = \hat{z}$

$\det(R) = 1$

~~left-handed coord: $\det(R) = -1$~~

det = 1 ↓ orth. ↙

$SO(3) = \{3\times 3 \text{ orthog. mat } R,\ \det(R) = 1\}$

= config space for rigid body w/ one pt fixed.

$R_{ij} = \delta_{ij}$:

$\delta_{ik}\delta_{ij} = \delta_{kj}$

Write $R_{ij} = \delta_{ij} + \underset{\text{infinitesimal}}{\varepsilon} T_{ij}$

$\cancel{\delta_{jk}} = (\delta_{ik} + \varepsilon T_{ik})(\delta_{ij} + \varepsilon T_{ij})$

$= \cancel{\delta_{jk}} + \varepsilon[T_{ik}\delta_{ij} + \delta_{ik}T_{ij}]$

$+ \varepsilon[T_{jk} + T_{kj}] + \dots$

$0 = T_{jk} + T_{kj}$

$T_{jk} = -T_{kj}$ (antisymmetr.)

$$T = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

Local coords: (a,b,c): 3-dimensional

3 What is the effective theory for the motion of a rigid body?

coords on config space: R_{ij} ! (but R is constrained)

Use Lagrange multiplier: (Λ) $L = \cdots + \mathrm{tr}(\Lambda(R^T R - 1))$

$+ \Lambda_{ij}(R_{kj} R_{ki} - \delta_{ji})$ → F_{ij}

count

Coords on config space: 9 R_{ij} − 9 Λ_{ij} = 0 ???

$F_{ij} = F_{ji}$

$R_{kj} R_{ki} = R_{ki} R_{kj}$

$\frac{1}{2}(\Lambda_{ij} + \Lambda_{ji}) F_{ij}$

$\frac{1}{2}(\Lambda_{ij} F_{ij} + \Lambda_{ji} \underbrace{F_{ji}}_{= F_{ij}})$

only symmetric part of Λ survives: $6 = 9 - 3$

$L(R_{ij}, \dot{R}_{ij}) = \cancel{A(R_{ij})} + \Lambda_{ij}(R \cdots)$

$+ \cancel{B_{ij}(R)\dot{R}_{ij}}$

$+ \frac{1}{2} \dot{R}_{ij} \dot{R}_{ik} K_{jk}$

minimal L for "free" rigid body rotation