PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 9

Euler's equations

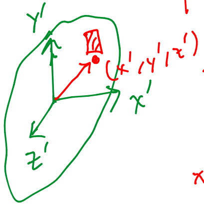
September 12

1 Derive microscopically the Lagrangian for rigid body rotation.

Last time: config space of rigid body = SO(3) [3×3 orth matrix R, $\det(R)=1$]

$$L = \underbrace{L_0[R_{ij}, \dot R_{ij}]}_{\text{kinetic energy } T} + \Lambda_{ij}\left(R_{ik}R_{jk} - \delta_{ij}\right)$$

Λ_{ij}: Lagrange multiplier



$$T = \int \underbrace{\frac{d^3x'}{2}\, \rho(x')}_{dm}\, v^2(x')$$

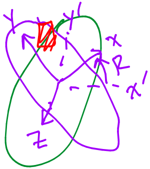
$x'_i = (x', y', z')$ fixed

$$v^2(x') = \partial_t x'_i\, \partial_t x'_i \quad [\partial_t \vec{x}' \cdot \partial_t \vec{x}']$$

$$= [\dot R_{ij} x_j][\dot R_{ik} x_k]$$

Jacobian $x' \to x$

$$\to \frac{1}{2}\int d^3x\, \det(R)\, \rho(x)\, \dot R_{ij} x_j \dot R_{ik} x_k$$



$$x'_i(x,t) = R_{ij}(t)\, x_j$$

"space frame"

x-points follow body (body frame)

$$T = \frac{1}{2} K_{jk} \dot R_{ij} \dot R_{ik}$$

$$K_{jk} = \int d^3x\, \rho(x)\, x_j x_k$$

"constant"

2 Write out the Euler-Lagrange equations.

$$L = \frac{1}{2} K_{jk} \dot{R}_{ij} \dot{R}_{ik} + \underbrace{\Lambda_{ij} (R_{ik}R_{jk} - \delta_{ij})}$$

$$K_{jk} = \int \frac{d^3x}{2} \rho(x)\, x_j x_k = x_k x_j = K_{kj}$$

$$\frac{\delta S}{\delta \Lambda_{ij}} = 0 = \frac{\partial L}{\partial \Lambda_{ij}} = R_{ik}R_{jk} - \delta_{ij}$$

$$\frac{\delta S}{\delta R_{ij}} = \underbrace{\frac{\partial L}{\partial R_{ij}}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{R}_{ij}}$$

$$\frac{d}{dt}\left[\frac{1}{2} \cdot 2 \dot{R}_{ib} K_{jb}\right] = \ddot{R}_{ib} K_{bj} = (\ddot{R} K)_{ij}$$

$$\frac{\partial}{\partial R_{ij}} \left[\Lambda_{ab} R_{ac} R_{bc}\right] = \dots = 2 \Lambda_{ib} R_{bj}$$

$$= \Lambda_{ab} \left[\frac{\partial R_{ac}}{\partial R_{ij}} R_{bc} + R_{ac} \frac{\partial R_{bc}}{\partial R_{ij}}\right]$$

$$\frac{\partial R_{ac}}{\partial R_{ij}} = \delta_{ai}\delta_{cj}$$

$$= \Lambda_{ab} \delta_{ai} \delta_{cj} R_{bc}$$

$$= \Lambda_{ib} R_{bj}$$

3 Get rid of the Lagrange multiplier.

$$\ddot{R}_{ik} K_{kj} = 2\Lambda_{ik} R_{kj}$$

$R_{\ell j}$ $\quad$ $R_{\ell j}$

$\ddot{R} K = 2\Lambda R \qquad [R^{-1} = R^T]$

$(\ddot{R} K R^T)_{i\ell} = (2\Lambda)_{i\ell}$

$\leftarrow R_{kj} R_{\ell j} = \delta_{k\ell} \qquad (RR^T = 1)$

$$\ddot{R}_{ik} K_{kj} R_{\ell j} = 2\Lambda_{i\ell}$$

Use: $\underline{2\Lambda_{i\ell} = 2\Lambda_{\ell i}}$

" $i\ell$ = ℓi "

$$\boxed{\ddot{R}_{ik} K_{kj} R_{\ell j} = \ddot{R}_{\ell k} K_{kj} R_{ij}}$$

$$\ddot{R} K R^T = R K^T \ddot{R}^T$$

"angular velocity" in space frame

Write $\dot{R}_{ik} = R_{ij} \Omega_{jk}$

$$\ddot{R}_{ik} = \dot{R}_{ij} \Omega_{jk} + R_{ij} \dot{\Omega}_{jk}$$

$$= R_{i\ell} \Omega_{\ell j} \Omega_{jk} + R_{i\ell} \dot{\Omega}_{\ell k}$$

$$= R_{i\ell} \left[(\Omega^2)_{\ell k} + \dot{\Omega}_{\ell k} \right]$$

4 What are the properties of Ω_{ij}?

$$\ddot{R}_{ik} K_{kj} R_{\ell j} = \ddot{R}_{\ell k} K_{kj} R_{ij}$$

$$R_{im}\left[\Omega_{mn}\Omega_{nk} + \dot{\Omega}_{mk}\right] K_{kj} R_{\ell j} = \text{same w/ } \ell, i \text{ swapped.}$$

$$R(\Omega^2 + \dot{\Omega}) \underbrace{R^T R}_{\text{ident}} K R^T$$

$$\tilde{K} = R K R^T$$

$$\tilde{\Omega} = R \Omega R^T$$

$$\left[\tilde{\Omega}^2 + \dot{\tilde{\Omega}}\right] \tilde{K} = \tilde{K} \left[\tilde{\Omega}^2 + \dot{\tilde{\Omega}}\right]^T$$

$$(\tilde{\Omega}^2 + \dot{\tilde{\Omega}}) \tilde{K} = \tilde{K} (-\tilde{\Omega})(-\tilde{\Omega}) - \tilde{K} \dot{\tilde{\Omega}}$$

$$\tilde{K} \dot{\tilde{\Omega}} + \dot{\tilde{\Omega}} \tilde{K} = \tilde{K} \tilde{\Omega}^2 - \tilde{\Omega}^2 \tilde{K}$$

Ω is antisymmetric (as is $\tilde{\Omega}$)

$$R(dt) = \underbrace{R(0)}_{\substack{\delta \\ \text{(ident.)}}}\left[1 + dt \underbrace{\Omega}_{\text{antisym.}}\right]$$

5 Derive Euler's equations.

$$\tilde{K}\dot{\tilde{\Omega}} + \dot{\tilde{\Omega}}\tilde{K} = \tilde{K}\tilde{\Omega}^2 - \tilde{\Omega}^2\tilde{K}$$

Choose body coords

$$\tilde{K} = \begin{pmatrix} K_1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_3 \end{pmatrix}$$

$$\tilde{\Omega} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

If $I_1 = K_2 + K_3$
$I_2 = K_3 + K_1$
$I_3 = K_1 + K_2$

Euler's Equations:

$$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3$$

$$I_2\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1$$

$$I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2 .$$

$$R(0)\underbrace{[1 + dt\,\Omega(0)]}_{t=dt}\underbrace{[1 + dt\,\Omega(dt)]}_{t=2dt}\ldots$$

$$\neq R(0)\left[1 + \int dt\,\Omega\right]$$