PHYS 5210: Graduate Classical Mechanics **Fall 2022**

Practice Exam

Problem 1 (Particle in a rotating frame): In this problem, we will study the Lagrangian and Hamiltonian mechanics of a non-relativistic point particle in three dimensions, moving in a rotating coordinate frame. A helpful way to do this is to say that $x_i(t) = (x_1(t), x_2(t), x_3(t))$ represents the particle's position in a non-rotating (inertial) frame, while $z_i(t)$ represents position in the rotating frame. These coordinates are related by

$$x_i(t) = R_{ij}(t) z_j(t), \tag{1}$$

where $R_{ij}(t) \in \mathrm{SO}(3)$ is a rotation matrix.

5 **A:** We assume in this problem that

$$\dot{R}_{ij} = R_{ik} \mathit{\Omega}_{kj}, \tag{2}$$

where the 3×3 matrix

$$\mathit{\Omega} = \omega \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3}$$

Argue that $R_{ij}(t)$ will be orthogonal.

15 **B:** Start with the Lagrangian

$$L = \frac{1}{2} \dot{x}_i \dot{x}_i. \tag{4}$$

(We are working in units where mass $m = 1$.)

B1. Plug in the ansatz (1) into (4) and show that

$$L = \frac{1}{2} \left(\dot{z}_i + \mathit{\Omega}_{ij} z_j \right) \left(\dot{z}_i + \mathit{\Omega}_{ik} z_k \right). \tag{5}$$

Hence we obtain the effective theory of a free particle, as viewed in a rotating (non-inertial) frame.

B2. Evaluate the Euler-Lagrange equations for $z_i(t)$. Explain how you can interpret the resulting equation, in the language of Newtonian mechanics, as motion in the presence of a "Coriolis force" and a "centrifugal force". (If you don't remember what these are, you can look them up online!)

10 **C:** Carry out the Legendre transform to find a Hamiltonian $H(p_i, z_i)$, where p_i is the canonical momentum for coordinate z_i.

10 **D:** Does a canonical transform from (p_i, z_i) to (P_i, Z_i) exist such that

$$H = \frac{P_1^2 + P_2^2 + P_3^2}{2} - \frac{\omega^2}{2} \left(Z_1^2 + Z_2^2 \right)? \tag{6}$$

Why or why not?

Problem 2: A positively charged particle of charge q and mass m moves in an electric potential $\Phi = \mathcal{E}_0 x$ ($\mathcal{E}_0 > 0$ is a constant). Assume its non-relativistic motion is restricted to the x-direction for simplicity.

10 **A:** Let us work out the Hamiltonian mechanics of this problem.

A1. Write down the Hamiltonian for this system.

A2. Suppose that we put a hard wall at $x = 0$, such that the dynamics is restricted to $x \geq 0$. Explain why the particle will now bounce back and forth forever, and calculate the period T of the motion. Assume that collisions conserve energy.

10 **B:** This problem can also be solved using action-angle variables.

B1. Describe how to define an action variable J for this problem. Then evaluate J in terms of $\mathcal{E}_0$, m, q, and the energy E of the particle.

B2. Use the action-angle formalism to calculate the period T of the motion, and check that your answer agrees with A2.

5 **C:** Suppose that the electric field $\mathcal{E}_0$ is slowly reduced. How will the energy E of the bouncing charged particle vary with $\mathcal{E}_0$?

Problem 3 (Superfluid vortex): Consider a relativistic field theory for a scalar field ϕ which is periodically identified with itself: namely we think of the following two configurations as the same:

$$\phi \sim \phi + 2\pi. \tag{7}$$

Such a field theory often describes the transition to superfluidity in condensed matter physics: the phase ϕ represents the "collective phase" of the wave function into which bosons are condensing.

15 **A:** Let us begin with the most general possible Lagrangian density $\mathcal{L}$.

A1. Write down invariant building blocks under Lorentz symmetry, along with (7), including at most two space/time derivatives.

A2. Show that if we demand further invariance under

$$\phi(x) \to \phi(x) + c, \tag{8}$$

then the unique Lagrangian (up to overall scale, and terms that would not affect equations of motion) is (for some constant $A > 0$) becomes

$$\mathcal{L} = -A\partial_\mu \phi \partial^\mu \phi. \tag{9}$$

A3. What are the Euler-Lagrange equations for the theory (9)?

A4. For $\mathcal{L}$ given in (9), calculate the stress energy tensor $T^{\mu\nu}$.

10 **B:** Suppose there are two spatial dimensions. Consider the field configuration

$$\phi(x, y) = \arctan \frac{y}{x} = \theta \tag{10}$$

Here θ is the angular coordinate in polar coordinates. This is called a **vortex**.

B1. Argue that $\phi(x, y)$ is a valid field configuration – at least away from the origin $x = y = 0$ – and that it solves the equations of motion found in A3.

B2. Argue, based on your answer to A4, that the vortex configuration has a "divergent" (infinite) energy.[1]

Your answer to B2 implies that vortices are "heavy" objects with slow dynamics, which will therefore play an important role in any effective theory of superfluidity.

Problem 4 (Center of mass conservation): In a Galilean-invariant world (such as our basically non-relativistic everyday world), the center of mass of a body does not move in its own rest frame. Suppose however that we mandated that this was the case in *all* frames, even one where the body had a finite momentum. We would then look for a many-body Hamiltonian H where both center of mass and momentum were conserved.

This is most elegant to do in the Hamiltonian formulation of mechanics. Consider a world with one space dimension, with particles $i = 1, \dots, N$ arranged on a line. Let x_i denote the displacement of particle i from equilibrium, and p_i its momentum. We wish to have

$$[H, P] = [H, X] = 0 \tag{11}$$

where X is the center of mass of the system, and P is the total momentum:

$$X = \sum_{i=1}^{N} x_i, \tag{12a}$$

$$P = \sum_{i=1}^{N} p_i. \tag{12b}$$

10 **A:** Let us begin by understanding the implications of the non-trivial Poisson brackets above. As discussed in Lecture 27, we should look for invariant building blocks under these symmetry transformations.

A1. Evaluate the Poisson brackets $[X, x_i]$, $[X, p_i]$, $[P, x_i]$, $[P, p_i]$.

A2. What are the invariant building blocks under the symmetries generated by X and P?

A3. Assuming spatial locality and homogeneity, we look for a Hamiltonian of the form

$$H = \sum_{i=1}^{N-1} h(x_i, x_{i+1}, p_i, p_{i+1}). \tag{13}$$

What is the most general form of h, given the building blocks from A2?

10 **B:** A minimal model for the h above is

$$h(x_1, x_2, p_1, p_2) = A\frac{(x_1 - x_2)^2}{2} + B\frac{(p_1 - p_2)^2}{2}, \tag{14}$$

where $A, B > 0$ are phenomenological constants. For this part only, consider $N \to \infty$.

B1. Find Hamilton's equations of motion for $\dot{x}_j$ and $\dot{p}_j$. What is their general solution?

B2. In the long wavelength limit, describe the dispersion relation for the propagating degrees of freedom. For this part you should consider $N \to \infty$.

5 **C:** Is there a natural Lagrangian effective theory for degrees of freedom $x_i(t)$, with both conserved center of mass and conserved momentum? Why or why not?

[1]In actual systems the divergence is cutoff by various finite size effects, but there are still significant effects.

This model represents a simple realization of a mixing between "multipolar" (center of mass) and space-time (translation) symmetries. This is a very active area of research in current theoretical physics.

Problem 5 (Higher-rank gauge theory): Consider a gauge theory consisting of a "mixed-rank" gauge field A_t and $A_{ij} = A_{ji}$. We demand the theory is invariant under the gauge transformations

$$A_t \to A_t + \partial_t \lambda, \tag{15a}$$

$$A_{ij} \to A_{ij} - \partial_i \partial_j \lambda. \tag{15b}$$

10 **A:** Let us first predict the most generic gauge theory compatible with these symmetries. Assume rotation/reflection symmetry, along with time-reversal symmetry.

A1. Write down a minimal set of gauge invariant objects, generalizing $F_{\mu\nu}$ from the standard (electromagnetic gauge theory). Deduce the most general Lagrangian one can write down, involving as few derivatives as possible, but also capturing the dynamics of all non-trivial degrees of freedom in the problem.

A2. Find the Euler-Lagrange equations and describe their general solution, assuming that $\mathcal{L}$ contains only quadratic terms in A_t and/or A_{ij}.

5 **B:** If we try to couple this theory to matter, the most natural kind of current we can couple to is a mixed rank current (J^t, J^{ij}), such that $\mathcal{L} = \cdots + J^t A_t + J^{ij} A_{ij}$, where $\cdots$ denotes your previous Lagrangian.

B1. Explain why this matter theory must have

$$\partial_t J^t + \partial_i \partial_j J^{ij} = 0. \tag{16}$$

B2. Show that this unusual theory would conserve both charge Q and dipole moment D^i, given by

$$Q = \int \mathrm{d}^d x \; J^t, \tag{17a}$$

$$D^i = \int \mathrm{d}^d x \; x^i J^t. \tag{17b}$$