**PHYS 5210: Graduate Classical Mechanics** **Fall 2023**

# Exam

**Due:** December 18 at 11:59 PM. Submit on Canvas.

You are allowed to refer to any course materials (including posted solutions), any books, and the Internet (e.g. Wikipedia, papers). **Do not collaborate** with any human; **do not solicit help** via PhysicsForums, Chegg, Quora or any similar website. You may ask the instructor alone for help in the form of clarifying questions. Please cite (in any reasonable way) any online resources you have used.

**Problem 1:** Consider a scalar field theory for a function $\phi(t, x, y, z)$ in four spacetime dimensions. Assume that the theory is invariant under rotations of the spatial coordinates $(x, y, z)$, as well as under flipping any of the coordinates ($t \to -t$ etc.).

10 **A:** Suppose the theory is invariant under the shift symmetry

$$\phi \to \phi + \lambda, \tag{1}$$

where $\lambda$ is a constant. Write down the most general possible Lagrangian $\mathcal{L}$ for the theory consistent with the above symmetries, keeping only terms that have two or fewer derivatives.

20 **B:** Now suppose that we instead demand that the theory is invariant under (1) where $\lambda(x, y, z)$ is now a function of the spatial coordinates. As in Lecture 17, we will add a dynamical gauge field $A_i$ which transforms as (here $i$ is an index for spatial coordinates only):

$$A_i \to A_i + \partial_i \lambda. \tag{2}$$

B1. Show that the most general two-derivative Lagrangian is (for constants $c_{1,2,3,4}$):

$$\mathcal{L} = c_1(\partial_t\phi)^2 - c_2(\partial_i\phi - A_i)^2 - c_3(\partial_i A_j - \partial_j A_i)(\partial_i A_j - \partial_j A_i) + c_4 \partial_t A_i \partial_t A_i \tag{3}$$

B2. Describe the normal modes of this theory as quantitatively as you can.

**Problem 2:** Consider a 2-dimensional isotropic harmonic oscillator, with Hamiltonian

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2}m\omega^2\left(x^2 + y^2\right). \tag{4}$$

10 **A:** Convert the $x$-oscillator and the $y$-oscillator separately into action-angle variables $(\phi_x, J_x)$ and $(\phi_y, J_y)$ respectively. Show that the Hamiltonian becomes[1]

$$H = \omega\left(J_x + J_y\right). \tag{5}$$

15 **B:** Notice that (5) depends only on $J_x + J_y$. It is tempting to say that this curious fact is a consequence of an underlying symmetry. For simplicity, set $m = \omega = 1$ to reduce clutter.

[1] You can use calculations from Lecture 29 as part of your answer, as long as you explain the setup of the calculation.

B1. Explain why $\{J_x, H\} = \{J_y, H\} = 0$, but this is not enough to restrict to $H$ to be a function of only $J_x + J_y$.

B2. Consider also the rotation symmetry around the $z$-axis. Include the generator $L_z$ of such rotations, and determine the full symmetry algebra generated by $J_{x,y}$ and $L_z$.[2]

B3. Is this symmetry algebra enough to enforce that $H$ only depends on $J_x + J_y$? Why or why not?

10 **C:** Suppose that the Hamiltonian (5) is modified to

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2}m\omega^2 (1 + \alpha t)\, x^2 + \frac{1}{2}m\omega^2 g(t) y^2. \tag{6}$$

If the dynamics starts at $t = 0$, and if $\alpha \ll \omega$, is it possible to find a function $g(t)$ such that the energy of the oscillator is *approximately* conserved as a function of time for all $t > 0$? Explain your answer.

**Problem 3 (Symplectic maps):** In Lecture 35, we studied the kicked rotor as a two-variable discrete map that originated from integrating Hamilton's equations with a time-dependent Hamiltonian. The kicked rotor map is just one example of a more general family of **symplectic maps**, which this problem will explore in more detail.

Suppose that we have a dynamical Hamiltonian system on two-dimensional phase space with canonical Poisson bracket $\{x, p\} = 1$, generated by (possibly $t$-dependent) Hamiltonian $H(x, p, t)$. In this problem, suppose that

$$H(x, p, t) = H(x, p, t + T), \tag{7}$$

i.e. the motion is periodic with period $T$. Let

$$x_n = x(nT), \tag{8a}$$
$$p_n = p(nT). \tag{8b}$$

15 **A:** The discrete map from $(x_n, p_n) \to (x_{n+1}, p_{n+1})$ has special properties that tell us the map is symplectic.

A1. First, argue that there are two $n$-independent functions $f$ and $g$ such that

$$x_{n+1} = f(x_n, p_n), \tag{9a}$$
$$p_{n+1} = g(x_n, p_n). \tag{9b}$$

A2. Show that the functions $f(x, p)$ and $g(x, p)$ must obey:

$$\{f, g\} = 1 \tag{10}$$

Give a physical interpretation of this result.

A3. Upon letting $x \to \phi$ and $p \to J$, show that (10) holds explicitly for the kicked rotor map from Lecture 35.

10 **B:** Now, let us generalize the linear stability analysis from Lecture 37 to these two-dimensional maps. Suppose that we have a fixed point $(x_*, p_*)$ of the map, and write

$$x_n = x_* + \delta x_n, \tag{11a}$$
$$p_n = p_* + \delta p_n. \tag{11b}$$

The general linearization of a map of the form (9) is

$$\begin{pmatrix} \delta x_{n+1} \\ \delta p_{n+1} \end{pmatrix} = \mathsf{M} \begin{pmatrix} \delta x_n \\ \delta p_n \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \delta x_n \\ \delta p_n \end{pmatrix}. \tag{12}$$

[2] *Hint:* Remember to include all functions needed so that the Poisson bracket of any two functions in the algebra is a linear combination of other functions!

B1. Plug in (12) into (10), and show that $\det(\mathsf{M}) = 1$.

B2. We can study a linearized symplectic map in a basis-independent way by studying the eigenvalues of $\mathsf{M}$ (assuming it is diagonalizable). Show that, if $\mathsf{M}$ is real-valued, that the solutions $\lambda_{1,2}$ to its characteristic equation $\det(\mathsf{M} - \lambda \cdot 1) = 0$ must take the form:[3]

$$(\lambda_1, \lambda_2) = \left(\mu, \mu^{-1}\right) \quad \text{or} \quad \left(\mathrm{e}^{\mathrm{i}\mu}, \mathrm{e}^{-\mathrm{i}\mu}\right) \quad \text{or} \quad \pm(1,1). \tag{13}$$

B3. Conclude that in the right basis, a linearized symplectic map will always look like one of the three standard forms below (here $\mu$ is some real number):

$$\mathsf{M}_1 = \left(\begin{array}{cc} \mu & 0 \\ 0 & \mu^{-1} \end{array}\right), \tag{14a}$$

$$\mathsf{M}_2 = \left(\begin{array}{cc} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{array}\right), \tag{14b}$$

$$\mathsf{M}_3 = \pm\left(\begin{array}{cc} 1 & 0 \\ \mu & 1 \end{array}\right). \tag{14c}$$

10 **C:** Find explicit Hamiltonian dynamical systems (you don't have to pick a $t$-dependent $H$, necessarily!), which lead to symplectic maps via the prescription of this problem. Find examples of maps with (at least one) fixed point, where linear stability analysis around said fixed point leads to all of the behaviors listed in **B**, for $\mu \neq 0$.

**Problem 4 (Elastic rods):** In this problem, we will develop a more systematic effective field theory for elastic rods (similar to Homework 1 but now using the effective field theory of solids developed in Lectures 18 and 19). Let us consider a thin elastic rod which can effectively be modeled as a "one-dimensional solid" embedded in two-dimensional space; for simplicity we focus only on static configurations and neglect time-dependence.

10 **A:** In Lecture 18, we built a solid Lagrangian using $\sigma^I(x_i)$; however, because this elastic rod is embedded in a higher-dimensional space, we will have to instead "invert" the picture and consider a Lagrangian theory for $x(\sigma)$ and $y(\sigma)$, which trace out the physical spatial position of each element $\sigma$ of the rod.

A1. Following Lecture 18, argue that we should demand four continuous symmetries for our Lagrangian $L(x, y, \dot{x}, \dot{y}, \ldots)$. What are they and why? Here we are using $\dot{x} = \partial_\sigma x$ and $\dot{y} = \partial_\sigma y$.

A2. Following Lectures 18 and 19, demand that the rod is in equilibrium in any of the configurations

$$x = x_0 + \sigma\cos\theta_0, \tag{15a}$$

$$y = y_0 + \sigma\sin\theta_0, \tag{15b}$$

for any constants $x_0, y_0, \theta_0$. Thus deduce that

$$E = \int \mathrm{d}\sigma\, \epsilon\left(\dot{x}^2 + \dot{y}^2 - 1\right) \tag{16}$$

is the most general "action" consistent with the symmetries (and that doesn't depend on higher derivatives), together with the requirement of equilibrium. Here $\epsilon$ is any non-negative function with a global minimum at $\epsilon(0) = 0$, which we will take to be the only minimum. In fact, the "action" here is nothing more than the overall energy $E$ of elastic deformations!

[3] *Hint:* Explicitly evaluate out this determinant in terms of $\lambda, a, b, c, d$ and then use the result of B1 to simplify things.

10 **B:** As written, this effective theory is sick. To see how, consider a rod of length $L$, with boundary conditions $x(0) = y(0) = y(L) = 0$ and $x(L) = (1-\delta)L$, for some $0 < \delta < 1$.

- B1. Describe infinitely many "solutions" $(x(\sigma), y(\sigma))$ that obey the desired boundary conditions, and also have $E = 0$, thus minimizing the overall energy.
- B2. Since this conclusion must not be physical, we must modify the effective theory. Using the philosophy of effective theory, write down the simplest possible correction to $E[x, y]$ that can fix the problem raised above.
- B3. Use your improved effective theory to argue for the energy-minimizing configurations of the elastic rod, given the "boundary conditions" above.
- B4. In what limit does the Lagrangian derived in B2 reduce to the Lagrangian on Homework 1? Why?