Homework 1

Due: September 8 at 11:59 PM. Submit on Canvas.

Problem 1 (Buckling of elastic rods): A thin elastic rod whose length L is much larger than the dimension(s) of its cross section can be approximated as a one-dimensional continuous object. In this problem we will study static deformations of this rod. Assuming that the rod deforms only in the xy-plane, and that in equilibrium the height $y(x) = 0$ (i.e. the rod lies along the x-axis for $|x| \le L/2$), we aim to deduce (using effective theory) an expression for the energy functional

$$E[y(x)] = \int_{-L/2}^{L/2} \mathrm{d}x \; \epsilon(y, y', y'', \ldots) \tag{1}$$

of the rod.[1] Here, $y' = \mathrm{d}y/\mathrm{d}x$. Assume that $E[y(x) = 0] = 0$.

A: Let us begin by studying the case where the rod is not under stress.

- A1. Explain why curves $y(x) = c$ (with c constant) are a global translation of the rod.
- A2. Explain why $y(x) = bx$ (for infinitesimal b) can be thought of as a global rotation of the rod.
- A3. It should not cost any energy to either translate or rotate the rod (or do both). Conclude using effective theory (a few sentences/lines of calculation should suffice) that the minimal phenomenological model for E becomes

$$E = \int \mathrm{d}x \, \frac{\alpha}{2} \left(y''\right)^2 . \tag{2}$$

In this model, $\alpha > 0$ is a phenomenological constant.

This Lagrangian can be derived microscopically from the theory of elastic solids as well. The point of using effective theory is that we can reach the same answer without carefully considering the forces and torques on each element of the solid – this approach is far more efficient!

B: When the rod is placed under compressive stress (you do *not* need to show this), the energy becomes

$$E = \int \mathrm{d}x \, \left[\frac{\alpha}{2} \left(y''\right)^2 - \frac{F}{2} \left(y'\right)^2\right] . \tag{3}$$

Here $F > 0$ is related to the force of compression acting on the rod. Evidently, this theory will have more than one derivative in the action, and so we will need to generalize the derivation from Lecture 1 to study this theory.

- B1. Show that the functional derivative of (1) is

$$\frac{\delta E}{\delta y(x)} = \frac{\partial \epsilon}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial \epsilon}{\partial y'} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial \epsilon}{\partial y''} . \tag{4}$$

[1] Note that this energy functional is not related to the energy found via Noether's Theorem in Lecture 3. E in this problem is analogous to the action.

B2. In the derivation of (4), explain why you need to fix both y and y' at $x = \pm L/2$.

20 **C:** Now, let us study the specific model (3).

C1. What are the Euler-Lagrange equations for $y(x)$?

C2. Thus deduce the energy-minimizing configuration $y_*(x)$ subject to the boundary conditions $y(\pm L/2) = 0$ and $y'(\pm L/2) = \pm a$ (for some constant a).

C3. Evaluate the energy on this configuration, and show that (for some constant K)

$$E[y_*] = Ka^2 \sin\left(L\sqrt{\frac{F}{\alpha}}\right). \tag{5}$$

C4. Deduce the value of the critical force F_c such that, if $F > F_c$, the rod will spontaneously buckle under the applied force.[2]

Problem 2 (Hyperbolic "central force" problem): In this problem we will construct an effective theory for a two-dimensional system invariant under the continuous symmetry:

$$x(t) \to x(t) + \epsilon y(t), \tag{6a}$$
$$y(t) \to y(t) + \epsilon x(t). \tag{6b}$$

20 **A:** Assume that $x > y$. Follow the method of discovering invariant building blocks in Lecture 2 to show that $x^2 - y^2$ and $\dot{x}^2 - \dot{y}^2$ are invariant building blocks under (6).

15 **B:** One Lagrangian made out of these invariant building blocks is

$$L = \frac{1}{2}m\left(\dot{x}^2 - \dot{y}^2\right) - V\left(\sqrt{x^2 - y^2}\right). \tag{7}$$

As in Lecture 2, we could look for coordinates, analogous to polar coordinates, that are better suited for this problem. The proper choice of coordinates turns out to be **hyperbolic coordinates**:

$$x = \rho \cosh \zeta, \tag{8a}$$
$$y = \rho \sinh \zeta. \tag{8b}$$

B1. Plug in (8) into (7) and show that[3]

$$L = \frac{1}{2}m\left(\dot{\rho}^2 - \rho^2\dot{\zeta}^2\right) - V(\rho) \tag{9}$$

B2. Use the ζ equation of motion to show that

$$Q = -m\rho^2\dot{\zeta} \tag{10}$$

is a conserved quantity (i.e. constant of motion).

B3. Show that the equation of motion for ρ can be expressed as

$$m\ddot{\rho} = -\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}\rho}, \tag{11}$$

where V_{eff} is an effective potential that depends on Q.

[2] *Hint:* When does the lowest energy configuration not have $a = 0$?

[3] *Hint:* Use that $\cosh^2\zeta = \sinh^2\zeta + 1$. The hyperbolic trigonometric functions are defined on Wikipedia, e.g.

10 **C:** Suppose that we further take

$$V(\rho) = k\rho \tag{12}$$

where $k > 0$ is a constant. For a generic initial condition where $Q \neq 0$, describe qualitatively how the particle will move in the xy-plane.

Problem 3 (Two-dimensional gravity): It was shown in 2016 that a theory of two-dimensional gravity in asymptotically AdS spacetime can be described by a point particle degree of freedom $T(t)$, which roughly traces out the boundary of a fluctuating spacetime domain (recall: a worldline of one particle traces out a curve in spacetime). This theory was shown to be invariant under the following nonlinear transformations (a, b, c, d are arbitrary constants):

$$T(t) \to \frac{aT(t) + b}{cT(t) + d}. \tag{13}$$

5 **A:** Show that the transformations in (13) can be generated by the following two continuous transformations:

$$T \to T + \epsilon_1, \tag{14a}$$

$$T \to T + \epsilon_2 T, \tag{14b}$$

and the discrete transformation

$$T \to \frac{1}{T}. \tag{15}$$

So it suffices to check invariance under these simpler transformations.

15 **B:** Use the ideas developed in Lecture 2 to generate invariant building blocks.

B1. Show that $\ddot{T}/\dot{T}$ is the simplest invariant under (14).

B2. Unfortunately, $\ddot{T}/\dot{T}$ is not invariant under (15). Fix the problem, and conclude that the lowest-derivative effective theory invariant under the full symmetry (13) is (here A is some constant):

$$S = -A \int \mathrm{d}t \left(\frac{\dddot{T}}{\dot{T}} - \frac{3\ddot{T}^2}{2\dot{T}^2} \right). \tag{16}$$

This is called the **Schwarzian action**.

Since this is just point particle mechanics, it is "easier" to quantize this theory of gravity, vs. a full-fledged theory of quantum gravity in four spacetime dimensions.