Homework 10

Due: November 10 at 11:59 PM. Submit on Canvas.

Problem 1 (External forces): Consider a non-relativistic particle with mass m, moving in two dimensions in the presence of both a time-independent potential energy $V(x,y)$ and a homogeneous t-dependent force $F(t)$ in the x-direction:

10 **A:** Show that the equations of motion for Hamiltonian

$$H = \frac{p_x^2 + p_y^2}{2m} + V(x,y) - F(t)x \tag{1}$$

reproduce Newton's Laws for this system.

20 **B:** Sometimes it is instructive to view this problem in a new coordinate system.

- B1. Find the generating function F_2 for a Type 2 canonical transformation to new coordinates (X, Y, P_X, P_Y) in which the "potential energy" (X, Y-dependent) terms in H are t-independent.
- B2. What are the new coordinates, expressed as functions of the old coordinates (and t)? Check explicitly, using Poisson brackets, that the transformation was canonical.
- B3. Obtain Hamilton's equations in the new coordinate system.
- B4. Solve the problem exactly if $V = 0$, and explain how to interpret your answer.

20 **C:** Let us now view this problem from the perspective of Hamilton-Jacobi theory.

- C1. Start with (1). What is the Hamilton-Jacobi equation for $S(x, y, t)$?
- C2. Use separation of variables, as much as you can, to solve the Hamilton-Jacobi equation for the special case when $V = 0$. Compare your result to B4.

15 **D:** For the remainder of the problem, consider $V(x,y) = 0$ and assume that $F(t) = F$ is a time-independent constant. While you should have found in C2 that the Hamilton-Jacobi equation is separable in these coordinates, it is not the *only* coordinate system in which this problem is separable. Consider the *spatial coordinates* a and b, defined implicitly via

$$x = \frac{b^2 - a^2}{2}, \tag{2a}$$

$$y = ab. \tag{2b}$$

- D1. Find a set of momenta p_a and p_b such that the transformation to coordinates (a, b, p_a, p_b) is canonical.
- D2. Show that the Hamiltonian becomes

$$H = \frac{p_a^2 + p_b^2}{2m\left(a^2 + b^2\right)} - \frac{F}{2}\left(b^2 - a^2\right). \tag{3}$$

15 **E:** Up to quadratures (i.e., up to possible integrals over single variables), solve the Hamilton-Jacobi equation, starting with the H given in (3).

Problem 2 (Separability): Consider Lagrangian

$$L = \frac{1}{2}a(u,v)\dot{u}^2 + \frac{1}{2}b(u,v)\dot{v}^2 - V(u,v). \tag{4}$$

10 **A:** Perform the Legendre transform and find the Hamiltonian.

10 **B:** Write the Hamilton-Jacobi equation. Find the most general form of $a(u,v)$, $b(u,v)$ and $V(u,v)$ such that the Hamilton-Jacobi equation can be solved by separation of variables in the form

$$S = -Et + W_u(u) + W_v(v). \tag{5}$$

15 **Problem 3 (Singular solutions):** Consider a Hamiltonian dynamical system on phase space $\mathbb{R}^{2n}$ with canonical coordinates (x_i, p_i), along with the abstract problem of trying to solve the Hamilton-Jacobi equation given an initial condition $S(x_i, 0) = S_0(x_i)$.

1. Show (by explicit calculation, if nothing else) that the following represents a solution to the Hamilton-Jacobi equation:

$$S(x_i, t) = \int_0^t \mathrm{d}s \left[P_i(s)\partial_s X_i(s) - H(P_i(s), X_i(s), s)\right] + S_0\left(X_i(s=0)\right). \tag{6}$$

Here $P_i(s)$ and $X_i(s)$ depend on t and x_i, but to avoid clutter we suppressed this dependence above, and we choose X_i and P_i to solve Hamilton's equations for mixed boundary conditions

$$X_i(s=t) = x_i, \tag{7a}$$
$$P_i(s=0) = \partial_i S_0(X_i(s=0)). \tag{7b}$$

This result implies that the Hamilton-Jacobi equation can in general be solved by the **method of characteristics**, following trajectories that solve Hamilton's equations!

2. Consider a free non-relativistic particle. Using your previous result, find an $S_0(x)$ such that the (unique) solution to the Hamilton-Jacobi equation $S(x,t)$ is singular for times $t > t_0$. Explain physically what is happening.