PHYS 5210: Graduate Classical Mechanics **Fall 2023**

Homework 11

Due: November 17 at 11:59 PM. Submit on Canvas.

30 **Problem 1 (Adiabatic compression of a gas):** Consider a molecule in a one-dimensional non-relativistic gas. The molecule has mass m and bounces back and forth between the walls of a container of length L. The length L is assumed to slowly (adiabatically) vary in time. While in this problem we focus on the dynamics of a single molecule, we can use its behavior to estimate what happens for the gas as a whole.

1. Find an adiabatic invariant for this problem as a function of the molecule's energy, E.
2. Deduce, as L changes, how the temperature T of the gas (which is proportional to E), will vary.
3. The pressure P in the gas is proportional to the average force per unit time applied to the left (e.g.) edge of the box, by the molecule bouncing off of it. How will P vary as a function of L?

Problem 2: Consider a system with phase space $\mathrm{S}^1 \times \mathbb{R}$, with canonical coordinates ϕ and L where $\{\phi, L\} = 1$. Now consider Hamiltonian

$$H = \frac{L^2}{2} + V(\phi), \tag{1}$$

where

$$V(\phi) = \min_n |\phi - 2\pi n|. \tag{2}$$

Assume that $\phi \sim \phi + 2\pi$ is a periodic coordinate – the form of $V(\phi)$ is written above just to emphasize that this potential is periodic.

15 **A:** Sketch the (ϕ, L) phase space, and sketch the trajectories on phase space generated by Hamilton's equations.

20 **B:** Attempt to solve this problem in action-angle coordinates. Show that you have to split the phase space into disconnected components in order to use action-angle variables, and explain why this happens (see Lecture 29). In each "patch" of phase space, find appropriate action variable J and use it to compute the period of the dynamics explicitly.

Problem 3 (Integrability in many-body systems): This problem will describe a curious set of differential equations that, rather remarkably: (*1*) turns out to be a Hamiltonian system, and (*2*) is integrable. The methods sketched in this problem form the starting point for a sophisticated theory of integrability in one-dimensional many-body systems.

Consider the set of differential equations

$$\dot{x}_n = x_n(x_{n+1} - x_{n-1}). \tag{3}$$

Here x_n are the degrees of freedom, and n is an arbitrary integer. You can either consider this set of equations on an infinite line, or one with periodic boundary conditions, where n is understood to be mod N. (e.g. $x_0 = x_N$, $x_1 = x_{N+1}$, etc.).

15 **A:** This is a Hamiltonian dynamical system, if we define the Poisson bracket to be

$$\{x_n, x_{n+m}\} = x_n x_m \cdot \begin{cases} m & m = \pm 1, \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

A1. Check that the Jacobi identity is obeyed with this Poisson bracket:

$$\{\{x_a, x_b\}, x_c\} + \{\{x_b, x_c\}, x_a\} + \{\{x_c, x_a\}, x_b\} = 0. \tag{5}$$

A2. Show that (3) follows from Hamiltonian mechanics, with the Hamiltonian

$$H = \sum_{n=-\infty}^{\infty} x_n, \quad \text{or} \quad \sum_{n=0}^{N-1} x_n. \tag{6}$$

20 **B:** One of the most elegant ways to understand integrability in a many-body system is to find a **Lax equation**. Let F be a *matrix* made out of the variables on phase space, and suppose that there is some other matrix B for which

$$\dot{F} = [F, B]. \tag{7}$$

B1. Show that (7) is obeyed if we take

$$F = \sum_n \left[x_n |n+1\rangle\langle n| + |n-1\rangle\langle n| \right], \tag{8a}$$

$$B = -\sum_n \left[(x_n + x_{n-1})|n\rangle\langle n| + |n-2\rangle\langle n| \right]. \tag{8b}$$

Here we are using bra-ket notation from quantum mechanics to denote the elements of real-valued $N \times N$ matrices (if N is finite). There is an auxiliary basis of states labeled by the integers $|n\rangle$.[1]

B2. Use (7) to show that

$$Q_m = \text{tr}\left(F^m\right) \tag{9}$$

is a conserved quantity. Here you should take $m = 2, 4, 6, \ldots$ to be even.

20 **C:** Let us now show that this system is in fact integrable.[2] Take N finite and even.

C1. Explain why if $m \leq N$, the Q_m are linearly independent functions.

C2. Show that the system is integrable by explicitly checking that $\{Q_m, Q_{m'}\} = 0$.

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[1] *Hint:* Calculate $\dot{F}|n\rangle$ and compare to $[F, B]|n\rangle$.

[2] *Hint:* I expect this part to require some mathematical creativity. My suggestion would be to start by thinking about "simple cases" and see if you tease out some general principles.