Homework 12

Due: December 1 at 11:59 PM. Submit on Canvas.

Problem 1 (Time-dependent perturbation theory): In Lecture 35, we saw that the kicked rotor is a chaotic Hamiltonian system with two-dimensional phase space, where chaos is enabled by a time-dependent Hamiltonian. The purpose of this problem is to understand some of the physics of driven integrable systems, by developing a time-dependent perturbation theory, generalizing Lectures 32 and 33.

Consider a one-dimensional system expressed in action-angle variables (ϕ_0, J_0), with Hamiltonian

$$H = H_0(J_0) + \epsilon H_1\left(\phi_0, J_0, t\right), \tag{1}$$

where the perturbation is periodic in time:

$$H_1\left(\phi_0, J_0, t\right) = H_1\left(\phi_0, J_0, t + \frac{2\pi}{\Omega}\right). \tag{2}$$

20 **A:** Suppose that

$$H_1(\phi_0, J_0, t) = \sum_{m,n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}(m\phi_0 - n\Omega t)} h_{mn}(J_0). \tag{3}$$

Follow Lectures 32 and 33 to show that the Type 2 canonical transformation to new action-angle variables that one should make is generated by $\phi_0 J + \epsilon S_1$ where

$$S_1(\phi_0, J, t) = \sum_{m,n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}(m\phi_0 - n\Omega t)} s_{mn}(J_0) \tag{4}$$

where for $(m,n) \neq (0,0)$ and $\omega_0 = \partial H_0/\partial J_0$,

$$s_{mn} = -\frac{h_{mn}}{\mathrm{i}(m\omega_0 - n\Omega)}. \tag{5}$$

Find the new Hamiltonian $H(J)$ to first order in ϵ.

20 **B:** Suppose there are integers n_0 and m_0 for which $h_{m_0 n_0} \neq 0$, and J_* such that

$$m_0\omega_0(J_*) = n_0\Omega. \tag{6}$$

B1. Explain why perturbation theory will break down if $J \approx J_*$.

B2. To remedy the issue, consider the Type 2 canonical transformation to $(\tilde{\phi}, \tilde{J})$ generated by

$$\tilde{S}(\phi_0, \tilde{J}, t) = \tilde{J}\left(\phi_0 - \frac{n_0}{m_0}\Omega t\right). \tag{7}$$

Find the coordinate transformation from $(\phi_0, J_0) \to (\tilde{\phi}, \tilde{J})$ and find the new Hamiltonian exactly.

B3. Suppose $\tilde{J} = J_* + \delta J$. Taylor expand the Hamiltonian to quadratic order in δJ. Assuming that m_0 and n_0 do not share any common divisors, that $h_{2m_0,2n_0} = h_{3m_0,3n_0} = \cdots = 0$, and that $h_{mn} = h_{-m,-n}$, show that up to an overall constant and subleading terms of order $\epsilon \cdot \delta J$,

$$H_{\text{new}} = \frac{1}{2}\omega_0'(J_*)\,\delta J^2 + 2\epsilon h_{m_0 n_0}\cos\left(m_0\tilde{\phi}\right) + t\text{-oscillating terms.} \tag{8}$$

20 **C:** Approximate that you can ignore the t-dependent terms in (8). Describe the trajectories in the phase space $(\tilde{\phi}, \tilde{J})$ near the breakdown of time-dependent perturbation theory. In particular, give a clear description for what happens when our action-angle perturbation theory fails, and any possible implications for the breakdown of integrability.

Problem 2 (Planetary orbits): Consider Earth and Jupiter in orbit around the Sun. For simplicity, we'll model this system by approximating that Earth/Jupiter is a planet of mass $M_{\mathrm{E/J}}$ in a circular orbit of *fixed* radius $R_{\mathrm{E/J}}$ around the Sun. As in reality, $R_{\mathrm{E}} < R_{\mathrm{J}}$.

20 **A:** We begin by writing down a toy Hamiltonian to describe this dynamics.

A1. Argue that the Hamiltonian describing this system can be described using canonically conjugate coordinates: $\{\theta_i, L_j\} = \delta_{ij}$ for $i = \mathrm{E}, \mathrm{J}$, with

$$H = \frac{L_{\mathrm{E}}^2}{2M_{\mathrm{E}}R_{\mathrm{E}}^2} + \frac{L_{\mathrm{J}}^2}{2M_{\mathrm{J}}R_{\mathrm{J}}^2} - \frac{GM_{\mathrm{E}}M_{\mathrm{J}}}{\sqrt{R_{\mathrm{E}}^2 + R_{\mathrm{J}}^2 - 2R_{\mathrm{E}}R_{\mathrm{J}}\cos(\theta_{\mathrm{E}} - \theta_{\mathrm{J}})}}. \tag{9}$$

A2. Under what conditions do we expect the periodic orbits of each planet to be nearly stable; i.e. that the system stays integrable?[1]

20 **B:** Assume that we are in a parameter regime where the dynamics is integrable and where each planet's orbit is only weakly perturbed by the other.

B1. Suppose that Earth/Jupiter have orbital periods $T_{\mathrm{E/J}}$ respectively. In terms of G, $M_{\mathrm{E/J}}$, $R_{\mathrm{E/J}}$, $T_{\mathrm{E/J}}$, find an approximate formula for the angular coordinate θ_{E} as a function of time t, using first-order perturbation theory. In your answer, keep only the first non-trivial correction in powers of $1/R_{\mathrm{J}}$.[2]

B2. Suppose that residents of Colorado are in the winter months when $0 \le \theta_{\mathrm{E}} \le \pi/2$. Use first-order perturbation theory, and the data in Table 1, to estimate the extent to which the duration of winters fluctuates in any given year on Earth. Note that $G \approx 6.7 \times 10^{-11}\ \mathrm{N \cdot m/kg^2}$.

	mass (kg)	orbit radius (m)	period (s)
Earth	6×10^{24}	1.5×10^{11}	3×10^{7}
Jupiter	2×10^{27}	8×10^{11}	4×10^{8}

Table 1: Data on the orbits of Earth and Jupiter. (Crude values for simplicity.)

[1] Actually, this problem is always integrable, but we could (at least somewhere in phase space) break integrability by adding more planets! However this would make the calculation more annoying without changing the basic idea or conclusions, so you only need to consider the effect of Jupiter! Still, answer the question as if you were uncertain of the global integrability of the problem.

[2] *Hint:* Use perturbation theory from Lecture 33. In what coordinate system is it easiest to solve for the dynamics exactly? Using those good coordinates, find an expression for θ_{E}, and thus deduce its time dependence to first order in G.

20 **Problem 3 (Bouncing ball):** Consider a non-relativistic bouncing ball, moving in Earth's gravitational field of strength g. It is bouncing off of a moving plate with "infinite mass", which oscillates at height

$$y_0(t) = a \cos \omega t. \tag{10}$$

When the ball bounces off of the plate, it does so elastically.

1. Find a set of two reasonable variables such that the dynamics of this problem can be captured by a recurrence relation between the two variables (a la Lecture 35). Find this recurrence relation.[3]

2. Show that you can express these equations in dimensionless variables, where

$$G = \frac{g}{a\omega^2} \tag{11}$$

   is the sole dimensionless parameter in the problem. Numerically simulate the discrete map from 1. Show that you find signatures of both integrability and chaos in different regions of phase space, and/or at different values of G.

3. In what regimes can you explain the emergence of integrability? Why?

[3] *Hint:* The answer will be a set of *implicit* equations (most likely).