Homework 13

Due: December 8 at 11:59 PM. Submit on Canvas.

Problem 1 (Newton's method): Newton's method is a classic numerical technique for finding the roots of a known function $g(x)$, i.e. the points x_* where $g(x_*) = 0$.

10 **A:** The algorithm goes like this: given a point x_n which is a proposed root, find a "better" root x_{n+1} by drawing a locally tangent straight line to the curve $g(x)$ at the point $(x_n, g(x_n))$, and finding the point x_{n+1} where that line intersects the x-axis. Show that this method leads to

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}. \tag{1}$$

20 **B:** Suppose (without loss of generality) that for small x,

$$g(x) \approx ax + bx^2 + \cdots . \tag{2}$$

Namely, $g(0) = 0$. Assume that $b \neq 0$.

- B1. Follow Lecture 37 and describe the dynamics of the map (1) for (sufficiently) small x_n. Show that if $a \neq 0$, the map will converge to the fixed point *very fast* – quantify exactly how fast the convergence will occur.
- B2. What happens if $a = 0$? Compare the two behaviors to what we saw in Lecture 37.

10 **C:** The result of **B** might make you think that the Newton's method map should always converge efficiently. Show that this is not the case, however; find a differentiable function $g(x)$ for which the map (1) will actually exhibit chaotic dynamics.

20 **Problem 2:** Consider the "curve" S, formed by the limiting process shown in Figure 1.

1. Show that if $\alpha < 1/3$, S will not self-intersect.
2. Taking $\alpha < 1/3$, show that the box dimension of S is

$$d = \max\left[1, \frac{\log 5}{\log(\alpha^{-1})}\right]. \tag{3}$$

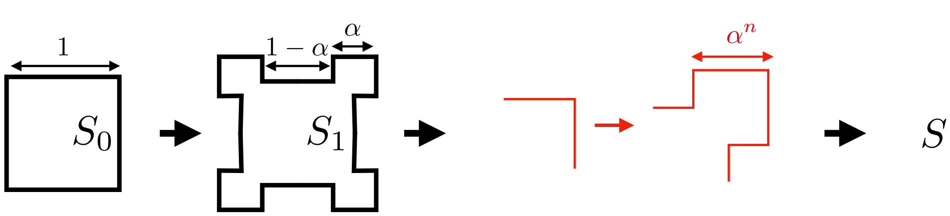

Figure 1: Start with S_0, a square of side length 1. At step n, take all corners and "add" the outer boundary of a square of length α^n, centered at the original corner, to make S_n. S is the limiting shape formed by this process.

3. Explain how it is possible that S appears "fractal" for any $\alpha > 0$, and yet $d = 1$ for $\alpha < 1/5$.

Problem 3 (Intermittency): Consider the logistic map from Lecture 36 with $r = 3.835$.

10 **A:** Use (if you want) the `Mathematica` package provided with Lecture 36. Simulate the dynamics for generic initial conditions and argue that there is a stable period-3 cycle.

15 **B:** Using a combination of analytics and numerics, we can explain this.

- B1. First, explain analytically that a stable period-3 cycle will require (at least) 3 stable fixed points of the map $f^{[3]}(x) = f(f(f(x)))$, with f the logistic map. What is the stability condition at these fixed points?
- B2. Numerically confirm that the logistic map has a stable period-3 cycle at $r = 3.835$.

15 **C:** It turns out that at $r = 3.828$ the stable period-3 cycle no longer exists.

- C1. Confirm this numerically, using a similar method to B2.
- C2. Locally, near a certain value of $\hat{x}$ (what does it correspond to?[1]), and near these values of r, the repeated logistic map locally looks like

$$f^{[3]}(x) \approx a(r - r_c) + x - b(x - \hat{x})^2 \tag{4}$$

  where r_c is the critical value just above which a stable period-3 cycle exists, and $a, b > 0$ are constants you don't need to determine. Use a cobweb diagram to argue that the logistic dynamics can get trapped near the period-3 cycle for a long time, when r is *just smaller* than r_c.
- C3. Numerically confirm by simulating the logistic map that this trapping behavior occurs. This is called **intermittency**, and it is responsible for the bursts of seemingly periodic behavior in chaotic dynamics that we saw in, e.g., Lecture 34.

10 **D:** Following Lecture 38, let us develop a renormalization group theory for intermittency. Recall that we look for a function $g(x)$ obeying

$$\alpha^{-1} g(\alpha x) = g(g(x)). \tag{5}$$

- D1. Argue that the universal function $g(x)$ that captures the intermittency transition should obey boundary conditions $g(0) = 0$ and $g'(0) = 1$.[2]
- D2. Show that the following ansatz solves (5) with $\alpha = 2$: for any constant c,

$$g(x) = \frac{x}{1 + cx}. \tag{6}$$

10 **E:** The function $g(x)$ found in (6) describes the universal behavior of a map *exactly* at the tangent bifurcation. However, as in **C**, let us consider perturbing the map by a small constant, such that we eventually pass through the region $x = 0$. This can be done by considering the map

$$g_r(x) = g(x) - r, \tag{7}$$

- E1. Treat r as a very small perturbation. Argue that $g_r(g_r(x)) \approx \alpha^{-1} g_{r'}(\alpha x)$ – what is r'?
- E2. Estimate the scaling of the number of steps $N(r) \sim r^{-\beta}$ it takes for the dynamics to pass through the intermittent region near an avoided tangent bifurcation. Namely, what is the value of β?

[1] *Hint:* Compare the answers to B2 and C1.
[2] *Hint:* Think about the "normal form" that we found above as $r \to 0$!