Homework 2

Due: September 15 at 11:59 PM. Submit on Canvas.

Problem 1 (Rotating reference frame): A non-relativistic particle of mass m moving in two dimensions is described by Lagrangian

$$L = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right) . \tag{1}$$

Consider viewing the particle from the perspective of a rotating reference frame, in which

$$x(t) = u(t) \cos(\omega t) + v(t) \sin(\omega t), \tag{2a}$$
$$y(t) = -u(t) \sin(\omega t) + v(t) \cos(\omega t). \tag{2b}$$

Here ω is a constant describing the rotation rate in the plane.

A: We can find the Lagrangian $L(u, v, \dot{u}, \dot{v})$ in the rotating reference frame by just *plugging in* to the original L in (1).

A1. Express $\dot{x}$ and $\dot{y}$ in terms of u, v, $\dot{u}$, $\dot{v}$.

A2. Hence, deduce an expression for $L(u, v, \dot{u}, \dot{v})$.

A3. Evaluate the Euler-Lagrange equations of motion for u and v, and identify the appearance of Coriolis and centrifugal "fictitious" forces.[1]

B: Show that the Lagrangian L in the rotating frame has time-translation symmetry. Is the resulting conserved quantity energy, as it would be defined in Newtonian mechanics? If not, explain why the discrepancy makes sense.

Problem 2 (Vortex dynamics): In this problem, we will use effective theory to build up a simple model for the dynamics of discrete (quantized) vortices in a superfluid. The elementary excitations are vortices that wind either counterclockwise ($\Gamma > 0$) or clockwise ($\Gamma < 0$).

A: We begin by deducing the effective theory for a single vortex. The degrees of freedom are the coordinates of a vortex (x, y). Assume that the theory should have translation invariance in x, y, and t, together with rotational invariance.

A1. Deduce that the most general Lagrangian capable of describing such a theory is

$$L = f \left(\dot{x}^2 + \dot{y}^2 \right) + \Gamma \left(x\dot{y} - y\dot{x} \right) . \tag{3}$$

where f is an arbitrary function and Γ is a constant. Since such symmetries were discussed in Lectures 2-4, you should quote appropriate results without re-deriving them.

[1] From the perspective of effective theory in the rotating frame, there is nothing "fictitious" about these effects!

A2. We do not usually write down the Γ term in (3) when building an effective theory for non-relativistic particles in two dimensions. Find a discrete symmetry, which might be reasonable to assume when thinking about ordinary particle dynamics, that the Γ term violates.

A3. Argue that this symmetry is violated by a vortex, and so the Γ term is allowed in this problem.

A4. As in Lecture 4, we often like to consider the limit of slow dynamics when building an effective theory. Conclude that we can approximate $f \approx 0$ in (3).

10 **B:** A theory for N interacting vortices is[2]

$$L = \sum_{i=1}^{n} \Gamma_i \left(x_i \dot{y}_i - y_i \dot{x}_i\right) - \sum_{i<j} \Gamma_i \Gamma_j V\left(\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}\right) \tag{4}$$

Here (x_i, y_i) denote the coordinates of vortex $i = 1, \ldots, N$; $\Gamma_i = \pm\Gamma$ denotes whether vortex i rotates clockwise or counterclockwise. The form of $V(r)$ is not important for this problem.

Find four conserved quantities from the continuous symmetries listed above.

15 **C:** Describe the dynamics of a pair ($N = 2$) of vortices. Show that, depending on the relative sign of Γ_1 and Γ_2, the vortex pair will either rotate in a circle or move in a straight line.[3]

Problem 3 (Molecular vibrations): Consider a molecule with M atoms labeled by $\alpha = 1, \ldots, M$. For simplicity, we will consider molecules that live in two spatial dimensions. Principles generalize naturally to three dimensions but calculations are (even) more tedious.

Let (x_α, y_α) denote the position of atom α. The Lagrangians that describe the molecular dynamics are of the form

$$L = \sum_{\alpha=1}^{M} \frac{m_\alpha}{2}\left(\dot{x}_\alpha^2 + \dot{y}_\alpha^2\right) - V, \tag{5}$$

where V is in general an extremely complicated function of the coordinates x_α and y_α (usually coming from a quantum Born-Oppenheimer calculation).

10 **A:** If our molecule doesn't interact with the rest of the (non-relativistic) universe, we expect that L has x, y and t translation invariance, rotation invariance in the xy plane, and Galilean boost invariance.

What are the requirements on m_α and V, if any, to ensure the theory has these symmetries?

15 **B:** Suppose that one static solution to the Euler-Lagrange equations – i.e., equilibrium – is

$$x_\alpha(t) = \bar{x}_\alpha, \quad y_\alpha(t) = \bar{y}_\alpha. \tag{6}$$

Following Lecture 5, we can Taylor expand V around this minimum. Writing

$$x_\alpha(t) = \bar{x}_\alpha + u_\alpha, \quad y_\alpha(t) = \bar{y}_\alpha + v_\alpha \tag{7}$$

and keeping terms at most quadratic in the perturbation strength, we obtain

$$V \approx \sum_{\alpha,\beta=1}^{M} \frac{1}{2} K_{\alpha\beta} q_\alpha q_\beta = \frac{1}{2} K_{\alpha\beta} q_\alpha q_\beta \tag{8}$$

[2] Because the repeated i index shows up three times, we write out the sum explicitly. The implicit summation convention makes sense so long as an index shows up exactly twice – otherwise it can be ambiguous.

[3] *Hint:* Use the results of part **B** liberally.

where $q_\alpha = (u_\alpha, v_\alpha)$ is shorthand for all the coordinates, and we have invoked the Einstein summation convention. The symmetries of part **A** require that the matrix $K_{\alpha\beta}$ has three null vectors.

Show that $K_{\alpha\beta}$ must have three linearly independent null vectors based on the continuous symmetries assumed above. Give explicit expressions for all of them.[4]

10 **C:** Consider the diatomic molecule H_2, discussed at the end of Lecture 3. Unlike in Lecture 3, now assume that the molecule can move in two spatial dimensions. You may assume without loss of generality that the equilibrium coordinates for the diatomic molecule are

$$\bar{x}_1 = -\bar{x}_2 = a, \tag{9a}$$
$$\bar{y}_1 = \bar{y}_2 = 0; \tag{9b}$$

here a is a constant parameter. Assume $m_1 = m_2$.

C1. Write down the most general choice of V consistent with part **B**.

C2. Argue that, as we claimed in Lecture 3, we can apply Noether's Theorem to reduce the dynamics of the molecule to a single dynamical degree of freedom.

15 **D:** Now consider the planar triangular molecule BH_3, as shown in Figure 1. Labeling the B with $\alpha = 4$ and the Hs with $\alpha = 1, 2, 3$, the equilibrium coordinates are (here b is a constant, and $m_1 = m_2 = m_3$):

$$\bar{x}_1 = 0, \tag{10a}$$
$$\bar{y}_1 = b, \tag{10b}$$
$$\bar{x}_2 = \frac{\sqrt{3}}{2} b, \tag{10c}$$
$$\bar{y}_2 = -\frac{b}{2}, \tag{10d}$$
$$\bar{x}_3 = -\frac{\sqrt{3}}{2} b, \tag{10e}$$
$$\bar{y}_3 = -\frac{b}{2}, \tag{10f}$$
$$\bar{x}_4 = 0, \tag{10g}$$
$$\bar{y}_4 = 0. \tag{10h}$$

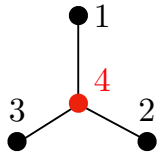

Figure 1: Sketch of the molecule BH_3.

D1. What are the three null vectors of $K_{\alpha\beta}$?

D2. Naively, the matrix $K_{\alpha\beta}$ would have five distinct eigenvalues, and we could not say anything about the eigenvectors of K. However, in this problem, the molecule BH_3 has some additional discrete symmetries that we can incorporate: rotation by $2\pi/3$, along with flipping $x \to -x$. *Carefully* deduce the resulting transformations on u_α and v_α.

D3. By demanding that $K_{\alpha\beta} q_\alpha q_\beta$ is invariant under the two transformations above, deduce that $K_{\alpha\beta}$ must have only 3 distinct eigenvalues.

This final calculation can be done elegantly using the mathematics of discrete group representation theory (but you can do it with brute force too). The key point is that we are able to make some rather non-trivial experimental predictions using very little knowledge of the complicated microscopics of this molecule. This is the philosophy underlying our effective theories!

[4] *Hint:* One of them is $u_\alpha = \bar{y}_\alpha$, $v_\alpha = -\bar{x}_\alpha$. Why?