PHYS 5210: Graduate Classical Mechanics **Fall 2023**

Homework 3

Due: September 22 at 11:59 PM. Submit on Canvas.

25 **Problem 1 (Massless relativistic particles):** In Lecture 6, we saw that the Lagrangian for a relativistic particle was proportional to mass m; however, there are also massless particles such as photons, and in principle a massless scalar particle could (in theory) also exist. In this problem we will explain how to build a Lagrangian for a massless particle.

1. Show that the equations of motion for the following Lagrangian:

$$L\left(x^\mu, \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}, \eta\right) = \frac{1}{\eta}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x_\mu}{\mathrm{d}\lambda} - \eta m^2, \tag{1}$$

   reproduce the Euler-Lagrange equation for x^μ for the theory described in Lecture 6.

2. Using (1), we can take the limit $m \to 0$ safely. Interpret the resulting Lagrangian. Does it make physical sense?

3. How should η transform under the reparameterization symmetry $\lambda \to f(\lambda)$, such that the action in terms of η and x^μ remains invariant under reparameterization?

Problem 2: Consider a charged relativistic particle of charge q and mass m, placed in a uniform electric field of strength E pointing in the x-direction. There are no magnetic fields.

15 **A:** Using the results of Lecture 7, show that we can write down a Lagrangian for this system:

$$S = \int \mathrm{d}\lambda \left[-m\sqrt{-\frac{\mathrm{d}x_\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}} - qEt\frac{\mathrm{d}x}{\mathrm{d}\lambda}\right]. \tag{2}$$

25 **B:** Some combination of the Lorentz transformations (rotations or boosts), together with spacetime translations, represent continuous symmetries of this problem.

- B1. Show that there are 6 continuous symmetries in this system.[1]
- B2. Use these symmetries, together with Noether's Theorem, to deduce the most general possible trajectory of the charged particle as a function of time.[2]

Problem 3 (Roller coaster): In this problem we will use Lagrangian mechanics to understand the design principles behind roller coasters. Assume for simplicity that the roller coaster behaves as a non-relativistic point particle of mass m constrained to move in the xy-plane along the track $y = h(x)$. The acceleration due to gravity has constant magnitude g and points in the $-y$ direction.

[1] *Hint:* There are 10 continuous symmetries total between spacetime translations and Lorentz transformations; if you can't immediately argue which 6 are symmetries, perhaps first find which 4 are *not* symmetries, and then check that whatever is left is in fact a symmetry.

[2] If you obtain formulas for the trajectory that depend on integrals over known functions that you can't evaluate analytically, you can express your answer in that form and get full credit.

15 **A:** Let us first consider a solution without Lagrange multipliers.

A1. Explain why (in the absence of the constraint) the Lagrangian for the particle motion is

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy. \tag{3}$$

You do not need to use effective theory arguments if you don't want to in this problem.

A2. Argue that you can set $m = 1$ without modifying the conclusions (but this will reduce some clutter in the calculation).

A3. Plug in the constraint $y = h(x)$ directly into L given in (3). Obtain a modified $L(x, \dot{x})$, and use the Euler-Lagrange equations to deduce the equation of motion for $\dot{x}$.

20 **B:** Now, we use Lagrange multipliers.

B1. Add a suitable Lagrange multiplier to (3) that enforces the constraint $y = h(x)$.

B2. What are the Euler-Lagrange equations for all degrees of freedom in the problem?

B3. Show how you can suitably combine these equations to obtain the same result as in A3.

B4. Argue that there is at least one benefit to the extra effort that went into using the Lagrange multiplier method: it allows you to easily calculate the normal force on the coaster. Give an explicit expression for this normal force.

15 **C:** Use the result of B4 to design a roller coaster loop. One simple design principle for a loop is that we would like the centripetal acceleration that the coaster feels to be constant as it traverses the loop. While this design principle does not fix a unique loop shape, explain how you can parameterize all possible loops that do obey this constraint, and make plots giving at least one example of such a shape. A numerical differential equation solver, such as `Mathematica`, will be needed for the last steps.