PHYS 5210: Graduate Classical Mechanics **Fall 2023**

Homework 4

Due: September 29 at 11:59 PM. Submit on Canvas.

Problem 1: Consider the rigid body rotation of a disk. Assume that, when the disk is oriented properly in space, the disk has both rotational symmetry about the z-axis, together with symmetry under reflections through the x, y and z axes. As discussed in Lecture 10, the Lagrangian for the theory will be of the form

$$L = \frac{1}{2}\dot{R}_{iI}\dot{R}_{iJ}K_{IJ} + \Lambda_{IJ}\left(R_{iI}R_{iJ} - \delta_{IJ}\right). \tag{1}$$

10 **A:** Argue that (for an appropriate choice of basis in the body frame) the symmetries of the disk imply

$$K_{IJ} \to \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & B \end{pmatrix} \tag{2}$$

for some constants A and B.

15 **B:** Let's see that (2) holds in a more microscopic model. Suppose that the rigid body is made up of N non-relativistic particles with coordinates $x_{i\alpha}$ ($\alpha = 1, \ldots, N$ denotes particle, while $i = 1, 2, 3$ denotes the spatial coordinates). Suppose that the Lagrangian for the microscopic particles is

$$L = \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha \dot{x}_{i\alpha}\dot{x}_{i\alpha}, \tag{3}$$

and that the microscopic particles are constrained to

$$x_{i\alpha}(t) = R_{iI}(t)\bar{x}_{I\alpha}, \tag{4}$$

where $\bar{x}_{I\alpha}$ is a fixed constant, and $R_{iI} \in \mathrm{SO}(3)$ is orthogonal.

B1. Plug in (4) into (3), and conclude that

$$K_{IJ} = \sum_{\alpha=1}^{N} m_\alpha \bar{x}_{I\alpha}\bar{x}_{J\alpha}. \tag{5}$$

B2. In the "continuum limit", we can approximate a rigid body to be a continuous object with mass density $\rho(\mathbf{x})$. We then replace (5) with

$$K_{IJ} = \int \mathrm{d}^3\mathbf{x}\; \rho(\mathbf{x}) x_I x_J. \tag{6}$$

Suppose that in cylindrical coordinates, a cylinder of mass M, radius R and height H has uniform mass density:

$$\rho(r, \theta, z) = \frac{M}{\pi R^2 H} \times \left\{ \begin{array}{ll} 1 & r < R,\ |z| \le H/2, \\ 0 & \text{otherwise} \end{array} \right. . \tag{7}$$

Calculate the 3 components of K_{IJ}, and confirm they are of the form (2).

15 **C:** Let us consider now the rigid body dynamics of our cylinder, assuming that $A \neq B$. Show that the Euler equations for a rigid body from Lecture 10 can be solved in complete generality for this system.[1]

Problem 2 (Global shape of SO(3)): In this problem, we will describe an alternative to the Euler angles of Lecture 11 that elucidates the global structure of SO(3).[2] Start with the Lagrangian $L = \frac{1}{2}\dot{R}_{iI}\dot{R}_{iJ}K_{IJ}$, and assume that $K_{IJ} = K_0\delta_{IJ}$ for simplicity.

25 **A:** Define the basis of antisymmetric 3×3 matrices

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

Suppose we choose $R(t) \in \mathrm{SO}(3)$ such that

$$R(t) = \exp\left[2\alpha(t)\left(\cos\beta(t)J_z + \sin\beta(t)\left[\cos\gamma(t)J_x + \sin\gamma(t)J_y\right]\right)\right]. \tag{9}$$

A1. Use `Mathematica` (or similar software for symbolic manipulation) to show that

$$L_0 = 4K_0\left(\dot{\alpha}^2 + \sin^2\alpha\dot{\beta}^2 + \sin^2\alpha\sin^2\beta\dot{\gamma}^2\right). \tag{10}$$

A2. Generalizing the discussion in Lecture 8, argue that the Lagrangian L_0 describes motion on the 3-dimensional sphere S^3, which is defined as the subspace of the 4-dimensional plane $(x, y, z, w) \in \mathbb{R}^4$ obeying $x^2 + y^2 + z^2 + w^2 = 1$. You can use `Mathematica` for algebraic manipulations, but your answer should clearly communicate the physics/math.

5 **B:** It turns out that the $R(t)$ parameterized above completely capture all possible SO(3) matrices, and thus our coordinates completely cover configuration space. However, there is something a little bit peculiar. Find a trajectory $[\alpha(t), \beta(t), \gamma(t)]$ that begins and ends at the same point in configuration space SO(3) – namely, the same R – yet does *not* begin and end at the same point on S^3. Use this construction to suggest that SO(3) must then be identified as S^3 with opposite points identified.

S^3 with opposite points identified is called three-dimensional real projective space $\mathbb{R}\mathrm{P}^3$. Alternatively, $\mathbb{R}\mathrm{P}^3$ is the set of all lines passing through the origin in $\mathbb{R}^4$.

Problem 3 (Nearly-rigid body motion): Consider two rigid bodies which are connected together to form a "nearly rigid body". Assume they rotate about a common fixed point. Let R_{iI} and S_{iI} denote the SO(3)-valued coordinates describing the configuration of each of the two rigid bodies.

15 **A:** An example of the Lagrangian you could write down for this dynamical system is

$$L = \frac{1}{2}A_{IJ}\dot{R}_{iI}\dot{R}_{iJ} + \frac{1}{2}B_{IJ}\dot{S}_{iI}\dot{S}_{iJ} + \frac{1}{2}C_{IJ}R_{iI}S_{iJ} + \Lambda_{IJ}\left(R_{iI}R_{iJ} - \delta_{IJ}\right) + \mu_{IJ}\left(S_{iI}S_{iJ} - \delta_{IJ}\right). \tag{11}$$

A1. Show that L is left-SO(3) invariant. Explain the physical meaning behind this mathematical requirement, how the desired symmetry should act on R and S, and why it is reasonable to assume in this problem.

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[1] You do not need to re-derive these equations from the Lagrangian; just assume the particular form of K_{IJ} and use results from lecture as needed.

[2] This problem is deeply connected to the *quantum* rotation group SU(2), which you can read about in the book (or learn in quantum mechanics).

A2. Give an example of a term that you could add to L without breaking the left-SO(3) invariance. Assume time-reversal symmetry, and do not multiply existing terms together!

15 **B:** Make the change of variables

$$S_{iJ}(t) = R_{iI}(t) Q_{IJ}(t). \tag{12}$$

B1. What is the Lagrangian in terms of R and Q? What are the constraints on Q?

B2. Justify the claim that writing L in terms of R and Q is more natural than in terms of R and S, from the perspective of effective theory, because Q will be a "fast" degree of freedom while R will be a "slow" degree of freedom.[3]

B3. When discussing molecular vibrations on e.g. Homework 2, you were able to *decouple* the slow and fast degrees of freedom in the Lagrangian. Is that decoupling possible here for our nearly-rigid body? As part of your answer, draw a picture that clearly explains (without equations) your conclusion.

20 **Problem 4 (Left-invariant dynamics on a Lie group):** In Lecture 10, we claimed that the most general Lagrangian with two derivatives for a rigid body was (up to Lagrange multiplier) $\dot{R}_{iI}\dot{R}_{iJ}K_{IJ}$. You may ask, however, why we did not write down the following even more general Lagrangian:

$$L = \frac{1}{2} A_{IJKL} R_{iI} \dot{R}_{iJ} R_{jK} \dot{R}_{jL} + \Lambda_{IJ} \left(R_{iI} R_{iJ} - \delta_{IJ} \right). \tag{13}$$

In this problem, we will explore the implications of this more general Lagrangian, where (unlike in class) R_{iI} will now be an $n \times n$ matrix.

As we will show, this Lagrangian is capable of describing dynamics on an abstract type of configuration space called a Lie group. For the purposes of this problem, you can think of an n-dimensional Lie group as characterized by a set of **structure constants**, which are fully antisymmetric tensors $f_{abc} = -f_{bac} = -f_{cba}$. Defining antisymmetric matrices T^a whose components obey $(T^a)^{bc} = f^{abc}$, a Lie group has the property that $\mathrm{tr}(T^a T^b) = \delta^{ab}$ and

$$[T^a, T^b] = f^{abc} T^c. \tag{14}$$

If $G = \mathrm{SO}(3)$, the structure constants $f^{abc} = \epsilon^{abc}$ are nothing but the Levi-Civita tensor. We see that rigid body rotation is the most important application of dynamics on Lie groups.

Generalizing Lecture 9, we can think of a configuration space for G by embedding it into the space of $n \times n$ matrices. Near the identity element of the group, you can think of the Lie group G as an n-dimensional space of the form (here ϵ_a is an n-dimensional vector)

$$R = 1 + \epsilon_a T_a + \cdots. \tag{15}$$

1. As in Lecture 10, define $\Omega_{IJ} = R_{iI}\dot{R}_{iJ}$. Show that there is an $n \times n$ antisymmetric matrix L_{IJ}, which you can interpret as "angular momentum", for which the equations of motion are

$$\dot{L} = [L, \Omega] = L\Omega - \Omega L. \tag{16}$$

2. Argue that we can treat the configuration space as the Lie group G if

$$A_{IJKL} = M_{ab} f_{aIJ} f_{bKL}. \tag{17}$$

Thus deduce the natural generalization of Euler's equations to arbitrary G.

3. Having derived the dynamics on more general Lie groups G, explain why our slightly simplified choice of Lagrangian in Lecture 10 captured the most general possible dynamics of three-dimensional rigid body rotation.

[3] *Hint:* The conclusion here is quite general, but you should use the explicit form of L given in (11) if it helps you.