Homework 5

Due: October 6 at 11:59 PM. Submit on Canvas.

Problem 1 (Wobbly Earth): The Earth is not a perfect sphere, nor is its mass fully isotropically distributed. As such, we can approximate Earth to be a rigid body which is symmetric about one axis: $I_1 = I_2$, and with

$$\frac{I_3}{I_1} - 1 \sim 3 \times 10^{-3}. \tag{1}$$

The axis the Earth rotates around most quickly is I_3: $\omega_3 \approx 1/(1 \text{ day})$.

15 **A:** Let us begin by discussing the solution to Euler's equations for a body of this kind. We have seen a similar problem on Homework 4. Plugging in for the actual numbers associated with Earth, describe the frequency of Earth's wobbling around the 3-axis in its body frame.

5 **B:** For the rest of this problem, we will consider a more non-trivial source of Earth's wobble, due to its gravitational interactions with the Sun. A microscopic calculation of the effect requires analyzing the gravitational potential energy of a non-spherical body, and one finds

$$L = \frac{I_1}{2}\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) + \frac{I_3}{2}\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 + \frac{3}{2}(I_3 - I_1)\omega_0^2\cos^2\theta \tag{2}$$

where $\omega_0 \approx 1/(1 \text{ year})$ is the orbital period of the Earth.

While fixing the exact prefactor of the last term above requires a microscopic calculation, explain why the function $\cos^2\theta$ that shows up in L is the minimal one which is consistent with either symmetries or physical principles relevant for the problem.

10 **C:** Suppose that the system is on a physical trajectory such that $\theta = \theta_0$ is independent of time.

- C1. Evaluate the Euler-Lagrange equations for L given in (2), and find an equation that constrains the value of θ.
- C2. Following Lecture 12, use the conserved quantity $p_\psi = I_3\omega_3$ to simplify your result from before to an equation relating θ and $\dot{\phi}$.
- C3. By using the physical values of ω_0, ω_3 and I_3/I_1, argue that the consistent solution to this equation has $\dot{\phi}$ very small. Estimate it, and thus the period of Earth's precession due to gravitational interactions with the Sun. Compare with the period from part **A**.

15 **D:** Follow our analysis of the spinning top in Lecture 12, and show that we can analyze the motion of Earth's wobble by mapping on to an auxiliary one-dimensional dynamical system, for a particle constrained to $-1 \le z \le 1$, with zero energy, and potential (per mass)

$$V_{\text{eff}}(z) = -\left(1 - z^2\right)\left(a + bz^2\right) + (c - dz)^2. \tag{3}$$

Give expressions for the constants a, b, c, d in terms of $I_1, I_3, \omega_0, \omega_3$.

E: Now consider more general a, b, c, d. You should ensure that at least in principle the values are physical (e.g. you do not set a parameter that must be positive to in fact be negative!); you can also assume that $I_3 > I_1$, as it is for Earth.

Qualitatively deduce all possible motions of a wobbly planet, by sketching all possible shapes for $V_{\text{eff}}(z)$ (focusing on the number of zeros and where V_{eff} is positive vs. negative).

Problem 2 (Nematic liquid crystals): In this problem, we will build an effective field theory for the dynamics of nematic liquid crystals. A nematic is a rod-like molecule, whose configuration space can be understood to be the space of lines passing through the origin of three-dimensional space. Similar to what we saw on Homework 4, this configuration space is a two-dimensional space called $\mathbb{RP}^2$, and we can think of it as the two-dimensional sphere S^2 with opposite points identified.

We will not really consider the full theory of a *liquid*, where the molecules can move relative to each other – we will assume that the nematic molecules are frozen in space, and focus on the rotational dynamics of the nematics relative to each other.

A: Given the configuration space described above, we can think of building an effective theory for the nematic by writing down $S[n_i(x_j, t)]$, where (x_j, t) denote the three spatial coordinates and time, while n_i denotes a unit vector on S^2.

A1. Assuming spacetime locality, argue that we should write down

$$S[n_i, \lambda] = \int \mathrm{d}^3x\mathrm{d}t \left[\mathcal{L}(n_i, \partial_t n_i, \partial_j n_i, \ldots) + \lambda(n_i n_i - 1)\right], \tag{4}$$

and explain the role of the λ term in this action.

A2. Why should we require that $\mathcal{L}(n_i, \partial_t n_i, \partial_j n_i) = \mathcal{L}(-n_i, -\partial_t n_i, -\partial_j n_i)$?

A3. Assume that the system has spacetime translation symmetry, and a combined spatial rotational symmetry under which we rotate the unit vector n_i and the spatial coordinate x_i together. Also assume that we have time-reversal symmetry $t \to -t$ and spatial inversion symmetry under $x_i \to -x_i$. Conclude that if we only allow for two derivatives in space and time, the most general Lagrangian density is[1]

$$\mathcal{L} = A\partial_t n_i \partial_t n_i - B(\partial_i n_i)^2 - C\partial_i n_j \partial_i n_j - D n_j(\partial_j n_i) n_k(\partial_k n_i). \tag{5}$$

B: In what follows, assume that $A, B, C, D > 0$. In equilibrium, the nematic liquid crystals are all aligned:

$$\bar{n}_i(x, t) = (0, 0, 1). \tag{6}$$

Consider small fluctuations around this equilibrium:

$$n_i = \bar{n}_i + \delta n_i, \tag{7}$$

with δn_i infinitesimally small.

B1. Find constraints on δn_i coming from the requirement that the dynamics stays on configuration space. In this part of the problem and what follows, keep only first order terms in δn_i.

B2. Write down the Euler-Lagrange equations for δn_i.

[1] You should neglect any terms that differ only by a total derivative.

B3. Show that these equations of motion are identical to those that you would have found by first plugging in the ansatz (7) into S, and keeping only quadratic terms in δn_i. Explain why this makes sense.

B4. Solve these equations, assuming that $\delta n_i \sim \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)}$. As part of your solution, you can try to orient any coordinate axes in a convenient way, but you must describe all possible solutions to the equations of motion. Thus, deduce the "normal modes" of a nematic liquid crystal.

15 **C:** Suppose that the nematics can interact with an external (static) magnetic field $B_{ij} = -B_{ji}$.[2] Magnetic fields break time-reversal explicitly, but we might expect that our effective theory is **covariant** under time-reversal and inversion, meaning that the theory is unchanged under a suitable modification of the external field:[3]

$$\mathcal{L}(n_i, \partial_t n_i, \partial_j n_i, B_{ij}) = \mathcal{L}(n_i, -\partial_t n_i, \partial_j n_i, -B_{ij}) = \mathcal{L}(-n_i, -\partial_t n_i, \partial_j n_i, B_{ij}). \tag{8}$$

C1. Keeping terms of at most two derivatives, write down the most general possible Lagrangian obeying (8).

C2. Describe how, if at all, the normal modes change. For simplicity, you may assume that the magnetic field is aligned along the z-direction, i.e. only $B_{xy} = -B_{yx} \neq 0$. Keep only the leading order terms as $k \to 0$ in the dispersion relation $\omega(k)$ to simplify the calculation as much as possible.

[2] We are using B instead of F, as in Lecture 7, to emphasize that it is just the magnetic field of interest.

[3] The semicolon denotes that B is not a dynamical field, but an important external parameter!