Homework 6

Due: October 13 at 11:59 PM. Submit on Canvas.

Problem 1 (Non-relativistic quantum mechanics): In this problem, we will build an effective field theory for single-particle non-relativistic quantum mechanics (i.e. the Schrödinger equation). The fields in the problem will be denoted as ψ and $\bar{\psi}$ – while ultimately we will want to think of these fields as complex conjugates, for this problem you should treat them as independent, complex-valued fields.

As in lectures, in this problem $ij\cdots$ indices run over spatial coordinates only, while $\mu\nu$ indices run over spacetime coordinates.

20 **A:** The symmetries that we will impose on our theory of quantum mechanics are: spatial rotation symmetry SO(3), spacetime translation symmetries, and lastly a U(1) phase relabeling symmetry similar to Lecture 17, under which, for some constants λ, q and $\hbar$:

$$\psi(x^\mu) \to \psi(x^\mu)\mathrm{e}^{\mathrm{i}\lambda q/\hbar}, \tag{1a}$$
$$\bar{\psi}(x^\mu) \to \bar{\psi}(x^\mu)\mathrm{e}^{-\mathrm{i}\lambda q/\hbar}. \tag{1b}$$

- A1. What are the invariant building blocks under the U(1) symmetry?
- A2. Which of these are invariant under translations and spatial rotations?
- A3. Keeping only the lowest non-zero power of both time and space derivative terms in the Lagrangian, and keeping only quadratic terms in ψ and $\bar{\psi}$, deduce that the (leading-order) effective field theory can be chosen to be[1]

$$\mathcal{L} = \frac{\mathrm{i}\hbar}{2}\left(\bar{\psi}\partial_t\psi - \psi\partial_t\bar{\psi}\right) - \frac{\hbar^2}{2m}\partial_i\psi\partial_i\bar{\psi} - \mu_0\bar{\psi}\psi. \tag{2}$$

  The form of $\mathcal{L}$ here is chosen for future convenience. So far, m, μ, $\hbar$ are phenomenological constants of the effective field theory. You do not need to explain the factor i in (2), but it is necessary for stability purposes.

10 **B:** Show that the Euler-Lagrange equations lead to the free particle Schrödinger equation.

20 **C:** For simplicity, now set $\mu_0 = 0$. Our Lagrangian is invariant under a number of spacetime symmetries, so we should expect a number of conserved currents. In what follows, use the results of Lecture 15.

- C1. Deduce a conserved current $J^\mu = (J^t, J^i)$ associated with the U(1) symmetry (phase rotation). What is its physical interpretation?
- C2. Deduce the energy current $J^\mu_{\mathrm{E}} = (\epsilon, J^i_{\mathrm{E}})$, where ϵ is the energy density and J^i_{E} is the spatial energy current/flux vector.
- C3. Deduce the *spatial* momentum density π^i and stress tensor/momentum flux T^{ij}.

[1] Note that you will have some arbitrary constant prefactors m, $\hbar$ etc. $\mathcal{L}$ has also been written in a particularly "symmetric" way, but your first guess might differ from this one by a total derivative term, in which case you will need to show how to obtain (2) as part of your answer.

C4. Are π^i and J^i_{E} related? Why or why not?

20 **D:** Let us try to gauge the U(1) symmetry, following Lecture 17. Although we will not show it in this problem, these manipulations are the starting point behind the theory of superconductivity in metals.

D1. Show that the λ transformation from C1 can be made coordinate-dependent if we incorporate a gauge field $A_\mu \to A_\mu + \partial_\mu \lambda$, and replace ∂_μ with a covariant derivative D_μ. Explain how you should choose D_μ to act on both ψ and $\bar{\psi}$.

D2. Now consider

$$\mathcal{L} = \frac{\mathrm{i}\hbar}{2}\left(\bar{\psi}\mathrm{D}_t\psi - \psi\mathrm{D}_t\bar{\psi}\right) - \frac{\hbar^2}{2m}\mathrm{D}_i\psi\mathrm{D}_i\bar{\psi} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{3}$$

Find the Euler-Lagrange equations from varying with respect to $\bar{\psi}$ and A_μ, and comment on their form in light of the results from part **C**.

15 **Problem 2:** In Lecture 16, we developed an effective field theory for relativistic electromagnetism. Explain what would happen if we relaxed the condition that the theory had to be invariant under Lorentz boosts. (Continue to demand invariance under translation and rotation.) Your answer should make sense as the effective field theory for electromagnetism in some ambient medium (like air or water), whose rest frame explicitly breaks the boost invariance of relativity.

Problem 3 (Burgers' equation): Burgers' equation describes the motion of an ideal velocity field in one spatial dimension:

$$\partial_t v + v\partial_x v = 0. \tag{4}$$

15 **A:** Show that this theory comes from a Lagrangian! By defining $v = \partial_x \phi$, show that the Euler-Lagrange equations for

$$\mathcal{L} = -\frac{1}{2}\partial_x\phi\partial_t\phi - f\left(\partial_x\phi\right) \tag{5}$$

reproduce (4) for a special choice of f. What is it?

15 **B:** Find an *infinite* number of conserved currents of this theory, together with the continuous symmetries responsible for them.

The infinity of conservation laws you (might have) found reveals that this theory is **integrable** – i.e., exactly solvable. We will discuss integrability a little more later in the class. There are a surprising number of integrable field theories in two spacetime dimensions, but very few interesting ones in higher dimensions (that have been discovered).