PHYS 5210: Graduate Classical Mechanics **Fall 2023**

Homework 7

Due: October 20 at 11:59 PM. Submit on Canvas.

Problem 1 (Crystal with cubic symmetry): In Lectures 18-20, we developed the formal effective field theory for an *isotropic* solid. This problem will describe how to do a similar construction for a solid made out of a crystal whose symmetry group is discrete. In particular, in this problem we will think about crystals made out of cubic lattices.

As we emphasized in Lecture 18, analogous to the theory of rigid body rotation, when building the effective field theory for a solid, we had to distinguish between the symmetries of how we can orient a solid in the "space frame", vs. the symmetries of the solid relative to itself. Within the effective field theory of a solid, what this means is that when a solid has an anisotropic crystal structure, the symmetry tensors associated with the reduced rotational group will (just like for a rigid body) be associated with "body frame" tensors invariant under that symmetry group.

25 **A:** In this problem, we consider a theory which is symmetric under the symmetry group of a cube: we can flip $\sigma^I \to -\sigma^I$ for any $I = 1, 2, 3$, and we can also swap σ^I and σ^J for any $I \neq J$.

A1. Show that the tensor

$$f^{IJKL} = \begin{cases} 1 & I = J = K = L \\ 0 & \text{otherwise} \end{cases} . \tag{1}$$

is invariant under the symmetries of the cube.[1]

A2. Argue that other than f^{IJKL} and $\delta^{IK}\delta^{JL}$, $\delta^{IJ}\delta^{KL}$, etc., there are no other invariants under the symmetries of the cube with four indices.

A3. Imposing the same symmetries as Lecture 18, argue that the leading order effective field theory for the solid is

$$\mathcal{L} = \frac{\rho}{2}\partial_t\sigma^I\partial_t\sigma^I - \frac{1}{8}\lambda^{IJKL}\left(\partial_i\sigma^I\partial_i\sigma^J - \delta^{IJ}\right)\left(\partial_k\sigma^K\partial_k\sigma^L - \delta^{KL}\right) \tag{2}$$

How many independent coefficients are there in the tensor λ^{IJKL}? What constraints on coefficients should we demand in order to ensure stability?

20 **B:** As in Lecture 19, plug in the ansatz $\sigma^I = \delta^I_i(x_i - \phi_i(x,t))$ into (2).

B1. What is $\mathcal{L}$, at quadratic order in ϕ_i and its derivatives?

B2. Calculate the stress tensor for this elastic solid. Express the final answer in terms of ϕ_i fields, not σ^I fields, and keep only the leading non-vanishing order.

B3. If you were an experimentalist, how would you measure all of the *independent* coefficients in λ^{IJKL}? Draw pictures to explain how you would apply strain and/or stress to the solid, being mindful of the intrinsic/preferred axes to the crystal with cubic symmetry.

[1] *Hint:* For convenience, you can think about whether $f^{IJKL}\sigma^I\sigma^J\sigma^K\sigma^L$ is invariant, if you like.

20 **C:** Following Lecture 20, we now study how sound waves propagate in a solid with cubic symmetry.

C1. Evaluate the Euler-Lagrange equations for the Lagrangian found in B1.

C2. Plug in the ansatz $\phi_i = a_i \mathrm{e}^{\mathrm{i}(k_j x_j - \omega t)}$, and find a set of algebraic equations that must be solved to find the dispersion relations of the normal modes.

C3. For simplicity, assume that $k_z = 0$. Now solve these equations analytically and describe the resulting normal modes of a solid. Compare to the isotropic solid discussed in Lecture 20.

Problem 2 (Fluids): In Lectures 18 and 19, we developed an effective field theory for (elastic) solid media. In this problem, we will discuss the *added constraints* that arise when studying the dynamics of a fluid. The crucial observation is that the difference between a solid and a fluid is that the latter will deform its shape to fill a container. Mathematically, one can reason that this means that the effective field theory for a fluid must have an additional symmetry that the solid does not have – invariance under **volume-preserving coordinate transformations**. This is a highly nonlinear symmetry group which allows for arbitrary transformations of the form:

$$\sigma^I(x) \to \xi^I(\sigma(x)) \tag{3}$$

for arbitrary functions $\xi^I(\sigma^1, \sigma^2, \sigma^3)$ obeying

$$\det\left(\frac{\partial \xi^I}{\partial \sigma^J}\right) = 1. \tag{4}$$

In addition to this symmetry, you should assume all of the symmetries of the isotropic solid from Lecture 18.

5 **A:** Begin by arguing (in a few lines) that with the postulated symmetry, it is indeed the case that a fluid could fill a container of arbitrary shape (but fixed volume) and stay in equilibrium.[2]

10 **B:** As we will see, this symmetry group is extremely constraining. As always, we need to begin by constructing invariant building blocks.

B1. Show that for any matrix $M^{\mu\nu}$, where here μ denote spacetime indices,[3]

$$\det\left(M^{\mu\nu}\partial_\mu\sigma^I\partial_\nu\sigma^J\right)$$

is invariant under the volume-preserving symmetry.

B2. Argue that under the symmetries of the isotropic fluid, there are 2 free parameters in the choice of $M_{\mu\nu}$. Deduce invariant building blocks that could enter into the Lagrangian:

$$\mathcal{L} = \mathcal{L}\left(\det\left(\partial_i\sigma^I\partial_i\sigma^J\right) - 1, \det\left(\partial_i\sigma^I\partial_i\sigma^J + c\partial_t\sigma^I\partial_t\sigma^J\right) - 1\right) \tag{5}$$

for some constant c. Why are the choices above nice?

20 **C:** As in Lecture 19, we expand the fluid around a background state: $\sigma^I = (x_i - \phi_i)\,\delta_i^I$.

C1. *Carefully* expand out any of the invariant building blocks to quadratic order in ϕ^I. Then deduce that at quadratic order, the most general Lagrangian describing a fluid is[4]

$$\mathcal{L} = A\partial_t\phi_i\partial_t\phi_i - B\left(\partial_i\phi_i\right)^2. \tag{6}$$

[2] *Hint:* Think about the function ξ^I as transforming a simple blob of volume V into the shape of a container of the same volume V.

[3] *Hint:* Use the chain rule to evaluate $\partial_\mu\xi^I(\sigma)$. Then, for two matrices A and B, $\det(AB) = \det(A)\det(B)$.

[4] *Hint:* For small ϵ, $\det(1+\epsilon A) = 1 + \epsilon\mathrm{tr}(A) + \frac{1}{2}\epsilon^2[\mathrm{tr}(A)^2 - \mathrm{tr}(A^2)]$.

C2. Describe the normal modes of a fluid, and compare them to those of a solid. Explain the most important difference between the two phases of matter, and relate the difference to the volume-preserving symmetry of a fluid.

15 **D:** Working in the linear response limit, where the ideal fluid Lagrangian is (6), *use Noether's Theorem* to show that for any curve γ in space,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\gamma} \mathrm{d}s_i \partial_t \phi_i. \tag{7}$$

Here the integral is a standard line integral over a three-dimensional vector field. Give a physical interpretation of this result.