PHYS 5210: Graduate Classical Mechanics **Fall 2023**

Homework 8

Due: October 27 at 11:59 PM. Submit on Canvas.

Problem 1 (Dualmon qubit): Consider the Hamiltonian

$$H = -\cos(2\pi q) - \cos\phi, \tag{1}$$

where $q \sim q + 1$ and $\phi \sim \phi + 2\pi$ are periodically identified coordinates, with Poisson bracket

$$\{q, \phi\} = 1. \tag{2}$$

The physical "realization" of this problem is in a superconducting circuit, where q and ϕ represent the accumulated charge on the edges of a nonlinear capacitor (called a quantum phase slip), while ϕ represents the magnetic flux through a nonlinear inductor (called a Josephson junction).[1]

10 **A:** Argue that the phase space is (topologically) a two-dimensional torus.[2]

20 **B:** Let us now analyze qualitatively the classical Hamiltonian dynamics of this system.

- B1. What are Hamilton's equations (of motion) for this system?
- B2. Sketch the trajectories of $q(t)$ and $\phi(t)$ in the two-dimensional phase space. You do not need to exactly solve the nonlinear equations of motion.
- B3. You should find that some of the trajectories look like straight lines that cross. Explain why, if two lines actually crossed, the dynamical system would be ill-posed. Explain why the dualmon equations of motion are well-posed, and resolve any apparent "paradox".

10 **C:** Show that the dualmon dynamical system is uniquely Hamiltonian – i.e. there is no Lagrangian $L(q, \dot{q})$ or $L(\phi, \dot{\phi})$ that correctly describes the dynamics of the dualmon on all of phase space.

There is an elegant theory for the Hamiltonian classical (and quantum) mechanics of superconducting circuits that we developed this year – see `arXiv:2304.08531`. Unfortunately, the dynamics you found above is not relevant to the quantum mechanics of the "dualmon", because quantum fluctuations turn out to be very strong when quantizing this Hamiltonian.

20 **Problem 2 (Vortex dynamics):** On Homework 2, we showed that N interacting vortices (labeled by $i = 1, \ldots, N$) in a two-dimensional superfluid (with circulations $\mathit{\Gamma}_i = \pm 1$) are described by Lagrangian[3]

$$L = -\frac{1}{2}\sum_{i=1}^{N} \mathit{\Gamma}_i \left(x\dot{y}_i - y\dot{x}_i\right) - \sum_{i<j} \mathit{\Gamma}_i \mathit{\Gamma}_j \log\sqrt{\left(x_i - x_j\right)^2 + \left(y_i - y_j\right)^2}. \tag{3}$$

Explain how to interpret vortex dynamics in the language of Hamiltonian mechanics. In particular, identify a suitable phase space, Poisson brackets, and Hamiltonian function H.

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[1] This particular superconducting circuit element is called the "dualmon", since it has both a nonlinear inductor and capacitor. The "transmon" qubit is much more famous, and has an ordinary capacitor in a loop with a Josephson junction.

[2] *Hint:* If we did not identify q but did identify ϕ, the phase space would be a cylinder. What if we now need to periodically identify $q \sim q + 1$?

[3] Note that we have changed a normalization factor for convenience in this problem.

Problem 3: Consider a particle moving on phase space $\mathbb{R}^4$ with coordinates (x, y, p_x, p_y) and canonical Poisson brackets.

10 **A:** Let $L_z = xp_y - yp_x$, and suppose that $\{L_z, H\} = 0$. Find the most general form of H compatible with this constraint – i.e. find all of the invariant building blocks.[4]

20 **B:** Now suppose that we also demand $\{p_x, H\} = 0$.

B1. Show how to evaluate $\{p_x, L_z\}$ explicitly (i.e. don't simply quote the answer from Lecture 23). Conclude from the Jacobi identity that if p_x and L_z are conserved, there must also be a third conserved quantity.

B2. Show that the three conserved quantities formed above form a Lie algebra under the Poisson brackets, and thus we can in principle find a Hamiltonian dynamical system with 3 conserved quantities.

B3. What is the most general Hamiltonian you can write down that has vanishing Poisson bracket with all 3 conserved quantities?

10 **C:** In part **A**, we saw the most general form of H obeying $\{L_z, H\} = 0$. Now, evaluate explicitly the Poisson bracket $\{p_x, H\}$ using this form of H, and show that H must actually be independent of *two* additional invariant building blocks, not just one. Hence, arrive at the same conclusion as part **B**, without (directly) invoking the requirement that the conserved quantities form a Lie algebra.

15 **Problem 4 (Lie algebras):** Consider two different realizations of a Lie algebra: one involving the Poisson brackets between a set of functions (as discussed in Lecture 23):

$$\{F^a, F^b\} = f^{abc} F^c, \tag{4}$$

and the other between a set of $n \times n$ matrices (for some integer n):

$$[\mathsf{M}^a, \mathsf{M}^b] = \mathsf{M}^a \mathsf{M}^b - \mathsf{M}^b \mathsf{M}^a = g^{abc} \mathsf{M}^c. \tag{5}$$

Do not make any assumptions about the M^a in this problem, besides assuming that the a index can only take a finite number of values.

Given a Lie algebra (5), can you always find a set of functions (and a Poisson bracket) which realizes (4), with $f^{abc} = g^{abc}$, on a finite dimensional phase space? Give a constructive proof, or find a counterexample to the claim.

[4] *Hint:* It might be easiest to guess the invariant building blocks and check they are all independent afterwards!