PHYS 5210: Graduate Classical Mechanics **Fall 2023**

Homework 9

Due: November 3 at 11:59 PM. Submit on Canvas.

Problem 1: Let (x, p) be canonically conjugate coordinates on $\mathbb{R}^2$. Let $\lambda \neq 0$ be constant. Consider the coordinate transformation

$$X = \lambda x, \tag{1a}$$

$$P = \lambda^{-1} p. \tag{1b}$$

10 **A:** Show that the transformation to (X, P) coordinates is canonical.

10 **B:** Find a function $F(x, p)$ that *generates* the canonical transformations parameterized by λ.[1]

20 **C:** As we saw in Lectures 24 and 25, the function F can be associated with a continuous symmetry of Hamiltonian H if $\{F, H\} = 0$.

- C1. What is the most general Hamiltonian $H(x, p)$ for which $\{F, H\} = 0$?
- C2. Show that you can solve Hamilton's equations of motion exactly for any Hamiltonian found in C1.[2] Does the answer make sense?
- C3. Find a function $G(x, p)$ for which $\{G, F\} = 1$.
- C4. Re-interpret your results from C1 and C2 in (G, F) coordinates. Do your previous results become easier to understand in these new coordinates?

Problem 2 (Rigid body rotation): In Lecture 26, we saw how to derive Euler's equations for a rigid body from Hamiltonian mechanics, using the body frame Poisson brackets

$$\{L_I, L_J\} = -\epsilon_{IJK} L_K. \tag{2}$$

Indeed, as discussed there, the dynamics can be reduced to first order equations for L_Is alone. Since there are only 3 L_I coordinates, these coordinates cannot by themselves be a "phase space" (i.e. a symplectic manifold). However, we can find a symplectic manifold by thinking a bit more carefully.

10 **A:** We have a "trial" phase space $\mathbb{R}^3$: (L_1, L_2, L_3). Argue that (2) alone restricts the dynamics to the space

$$L_0^2 = L_1^2 + L_2^2 + L_3^2 \tag{3}$$

where L_0 is a *constant*. Deduce that the physical phase space is the two-dimensional space found above. What is the geometry of this space?

[1] *Hint:* Let $\lambda = 1 + \epsilon$, with ϵ infinitesimal. Then compare to Lectures 24 and 25.

[2] *Hint:* Use Noether's Theorem to simplify the calculation, when appropriate!

10 **B:** Suppose we choose coordinates

$$\begin{aligned} L_1 &= L_0 \sin\theta \cos\phi, \quad (4a) \\ L_2 &= L_0 \sin\theta \sin\phi, \quad (4b) \\ L_3 &= L_0 \cos\theta. \quad (4c) \end{aligned}$$

Show that

$$\{\theta, \phi\} = \frac{1}{L_0 \sin\theta} \quad (5)$$

reproduces the angular momentum algebra (2).

20 **C:** We see a physical setting in which a *compact* symplectic manifold can arise in Hamiltonian mechanics. Let's show that the formalism developed in Lectures 24 and 25 continues to make sense.

C1. Describe the canonical transformation generated by the function L_3.[3] What is its physical interpretation?

C2. Suppose that we are given a Hamiltonian H for which $\{L_3, H\} = 0$. Describe the most general possible motion on phase space.

C3. Under what circumstances is this dynamics (and symmetry) realized by a Hamiltonian of the form

$$H = \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3}. \quad (6)$$

Does your result make sense?

Problem 3 (Symmetry algebras): In this problem, we will generalize our understanding of Noether's Theorem in Hamiltonian mechanics, building off of Lectures 23 and 25. We will (ultimately) see that in Hamiltonian mechanics, it is natural to consider a symmetry algebra generated by a set of functions F^a and H which are closed under the Poisson bracket:

$$\begin{aligned} \{F^a, H\} &= M^{ab} F^b + C^a H, \quad (7a) \\ \{F^a, F^b\} &= f^{abc} F^c + K^{ab} H. \quad (7b) \end{aligned}$$

In this problem, assume that F^a and H do not depend explicitly on time. You should further assume $\omega_{\alpha\beta}$ is the (ξ-independent) canonical symplectic form for the calculation.

20 **A:** Begin by assuming $C^a = K^{ab} = M^{ab} = 0$ – namely, we consider the same kind of symmetry algebras as Lecture 25, which leave the Hamiltonian invariant.

A1. Starting with (recall Lecture 21)

$$S = \int \mathrm{d}t \left[\frac{1}{2}\omega_{\alpha\beta}\xi^\alpha \dot{\xi}^\beta - H(\xi) \right], \quad (8)$$

use the Lagrangian formulation of Noether's Theorem (Lecture 3) to show that time-translation invariance implies H is a Noether charge.

A2. Show that F^a are conserved quantities (i.e. Noether charges) for the continuous symmetries generated by the functions F^a (as in Lecture 24 and 25).

[3] *Hint:* It might be useful to think about the coordinates from **B**.

10 **B:** Now consider the more general case, in which H itself can be included in the symmetry algebra. For technical convenience, go ahead and set $C^a = 0$.

B1. Again, consider the remaining symmetries associated with F^a. In this case, we need to think carefully about the correct symmetry transformations on the coordinates ξ^α. Argue that

$$\xi^\alpha \to \xi^\alpha + \epsilon\{\xi^\alpha, F^a(-t)\}. \tag{9}$$

B2. Show that (8) is invariant under (9); this requires a few careful manipulations. What is the corresponding Noether charge?

10 **C:** A physically relevant example of a theory with a non-trivial symmetry algebra of the kind studied in **B** is the theory of a particle in one dimension with Galilean boost invariance (and translation symmetries):

$$H = \frac{p^2}{2m}. \tag{10}$$

C1. What is the generating function of the canonical transformation corresponding to Galilean boosts?

C2. Derive the "Galilean symmetry algebra" in one dimension.

Although Hamiltonian mechanics is more abstract, it can have a much more elegant treatment of symmetries and the structure they imbue into a problem!