PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 10
Euler's equations

September 20

Configuration space of rigid body rotation: $SO(3)$ [3×3 matrices, orthogonal $RR^T = 1$, $\det = 1$]

Physically: matrix R is rotation from body frame (I) $\longrightarrow$ space frame (i)

$$R_{iI} : \left(\sum_{I=1}^{3}\right) R_{iI} R_{jI} = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}. \qquad R_{iI} R_{iJ} = \delta_{IJ}$$

Lagrangian for rigid body dynamics:

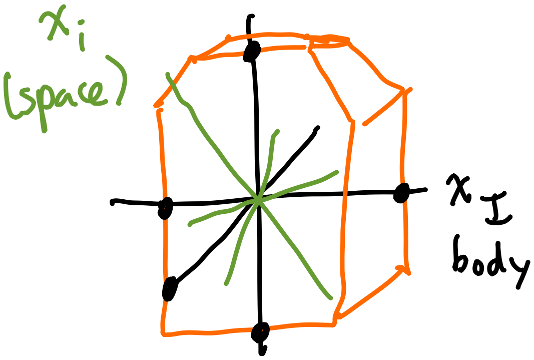
$$L = f(R_{iI}, \dot{R}_{iI}, \cdots) + \Lambda_{IJ}(R_{iI}R_{iJ} - \delta_{IJ})$$

Lagrange multiplier: $\Lambda_{IJ} = \Lambda_{JI}$.

Use effective theory to constrain f:

① Symmetries $\longrightarrow$ ② invariant BBs $\longrightarrow$ ③ EOMs

① Symmetries:



A rigid body may appear asymmetric in body frame

No preferred orientation in space.
(Choice of "origin", or "identity matrix", is arbitrary)

Physical: relative rotation over time.
Intuitively: like translation? (no preferred $x=0$), but on SO(3)?

Define: left SO(3) invariance: $S[R_{iI}(t)] = S[Q_{ij} R_{jI}(t)]$ (orthogonal/SO(3))
for any constant Q_{ij}. [Generalizes: $x(t) \to x(t)+\varepsilon$]

Contrast: w/ right SO(3): $S[R_{iI}(t) U_{IJ}] \overset{?}{=} S[R_{iJ}(t)]$
NOT symmetry; unless object is sphere.

Symmetries of rigid body rotation:
- left SO(3) invariance
- time-reversal symmetry
- right symmetry: of object

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② Invariant BBs:

Under time-reversal: $t \to -t$

$\dot{R}_{iI} \to -\dot{R}_{iI}$

$\downarrow$

2 time derivatives in L

Under left SO(3) invariance:

invariants of SO(3): $\underline{\delta_{ij}}$, $\cancel{\varepsilon_{ijk}}$ not use

Analogy to relativity: $\eta_{\mu\nu}$ invariant under Lorentz.

$\eta_{\mu\nu} a^\mu b^\nu$ invariant

→ contract indices w/ δ_{ij}: (no dangling space frame indices)

$$\underbrace{\cancel{R_{iI} R_{iJ}}}_{\delta_{IJ} = \text{const.}}$$

not dynamical

$$\underbrace{R_{iI} \dot{R}_{iJ}}_{\Omega_{IJ}}$$

$$\underbrace{\dot{R}_{iI} \dot{R}_{iJ}}_{\text{nice invariant.}}$$

Under right symmetry (body frame symmetry):

Today: no symmetry in body frame,

$$L = \frac{1}{2} \dot{R}_{iI} \dot{R}_{iJ} \underbrace{K_{IJ}}_{\text{Symmetric matrix}} + \Lambda_{IJ} (R_{iI} R_{iJ} - \delta_{IJ})$$

has manifest left-SO(3) invariance.

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③ EOMs:

$$\frac{\delta S}{\delta \Lambda_{IJ}} = 0, \quad \text{or} \quad R_{iI} R_{iJ} = \delta_{IJ}$$

$$\frac{d}{dt}(R_{iI} R_{iJ}) = 0$$

$$= \dot{R}_{iI} R_{iJ} + R_{iI} \dot{R}_{iJ}$$

$$= R_{iJ} \dot{R}_{iI} + R_{iI} \dot{R}_{iJ} = \underbrace{\Omega_{JI} + \Omega_{IJ}}_{\substack{\text{antisymmetric} \\ \Omega = -\Omega^T}} = 0$$

$$\dot{R}_{iJ} = R_{iI}\,\Omega_{IJ} \;:\; \Omega_{IJ} \text{ represents angular velocities relative to body frame!}$$

$$\frac{\delta S}{\delta R_{iI}} = 0 = -\frac{d}{dt}(\dot{R}_{iJ}K_{IJ}) + \Lambda_{IJ}R_{iJ}$$

$$\ddot{R}_{iJ}K_{IJ}R_{iK} = \Lambda_{IK}$$

$$\ddot{R}_{iJ} = \tfrac{d}{dt}(R_{iL}\Omega_{LJ}) = \dot{R}_{iL}\Omega_{LJ} + R_{iL}\dot{\Omega}_{LJ}$$

So: $$\ddot{R}_{iJ}K_{IJ}R_{iK} = \underline{\Omega_{KL}\Omega_{LJ}K_{IJ} + \dot{\Omega}_{KJ}K_{IJ} = \Lambda_{IK}}$$

take antisymmetric part!

Get rid of $\Lambda_{IK} = \Lambda_{KI}$ by using symmetric:

$$0 = \dot{\Omega}_{KJ}K_{IJ} - \dot{\Omega}_{IJ}K_{KJ} + \Omega_{KL}\Omega_{LJ}K_{IJ} - \Omega_{IL}\Omega_{LJ}K_{KJ}$$

Reduced dynamical system to body frame only. [left SO(3) invariance]

Define angular momenta:

$$L_{IJ} = -L_{JI} = K_{IK}\Omega_{KJ} + \Omega_{IK}K_{KJ}$$

$$\left[L = K\Omega + \Omega K\right]$$

sym antisym

Show: $$\dot{L}_{IJ} = L_{IK}\Omega_{KJ} - \Omega_{IK}L_{KJ} = [L,\Omega]_{IJ}$$

Write out:

angular velocity around 3-axis of body frame

$$L_{IJ} \to \begin{pmatrix} 0 & -L_3 & L_2 \\ L_3 & 0 & -L_1 \\ -L_2 & L_1 & 0 \end{pmatrix} \qquad \Omega_{IJ} \to \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

$\dot{L} = [L, \Omega] \longrightarrow$ Euler's equations:

$$\boxed{\begin{aligned} \dot{L}_1 &= L_2 \omega_3 - L_3 \omega_2 \\ \dot{L}_2 &= L_3 \omega_1 - L_1 \omega_3 \\ \dot{L}_3 &= L_1 \omega_2 - L_2 \omega_1 \end{aligned}}$$

Convenient to choose basis in body frame where K_{IJ} is diagonal.

$$K = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}$$

Define (principal) moments of inertia:

$$I_1 = K_2 + K_3$$
$$I_2 = K_3 + K_1$$
$$I_3 = K_1 + K_2$$

$$\left.\begin{aligned} L_1 &= I_1 \omega_1 \\ L_2 &= I_2 \omega_2 \\ L_3 &= I_3 \omega_3 \end{aligned}\right\} \quad L = K\Omega + \Omega K$$

Euler:

$$I_1 \dot{\omega}_1 = (I_2 - I_3)\omega_2 \omega_3$$
$$I_2 \dot{\omega}_2 = (I_3 - I_1)\omega_3 \omega_1$$
$$I_3 \dot{\omega}_3 = (I_1 - I_2)\omega_1 \omega_2 .$$