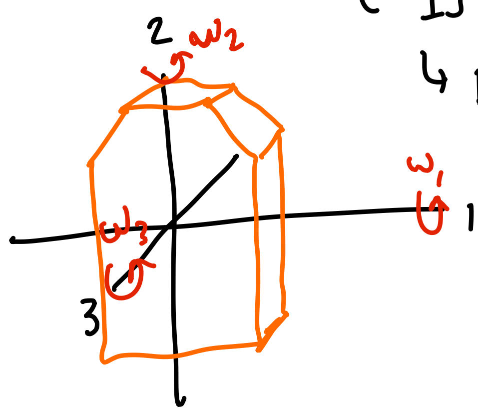
PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 11
Euler angles

September 22

$(K_{IJ}\dot{R}_{iI}\dot{R}_{iJ})$

↳ $K_1+K_2=I_3$



Euler equations:

$$I_1\dot{\omega}_1 = (I_2-I_3)\,\omega_2\omega_3$$
$$I_2\dot{\omega}_2 = (I_3-I_1)\,\omega_1\omega_3$$
$$I_3\dot{\omega}_3 = (I_1-I_2)\,\omega_1\omega_2$$

rotation rates in body frame

How to solve? Identify conserved quantities.

- energy

$$E = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$$
$$= \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3}$$

- (angular momentum)^2

$$= L^2 = (I_1\omega_1)^2 + (I_2\omega_2)^2 + (I_3\omega_3)^2$$
$$= L_1^2 + L_2^2 + L_3^2$$

Space frame

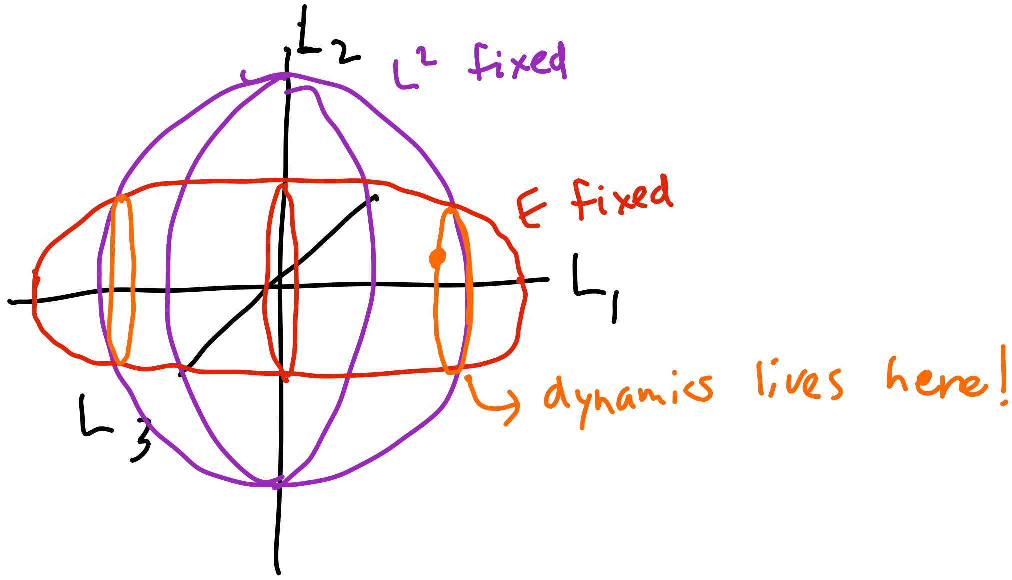
[L_i^2 = conserved

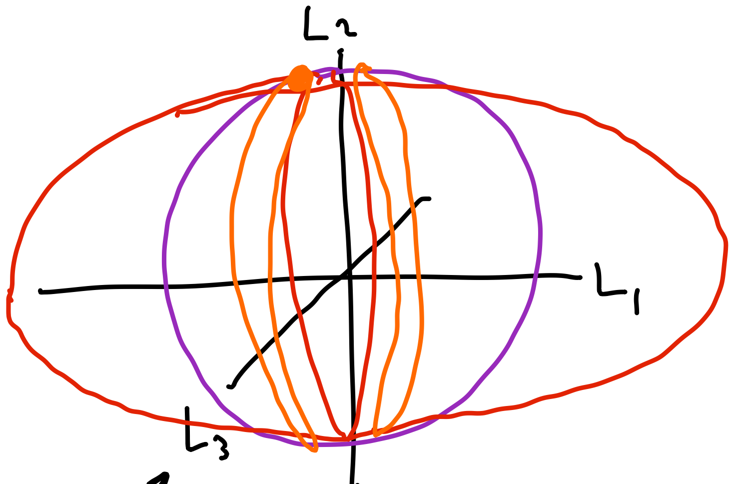
$= R_{iI}L_I$:

$L_iL_i = L_IL_I$]

2 constants of motion → dynamics in (L_1, L_2, L_3) constrained to intersection of 2 surfaces → 1d subspace.

Assume $I_1 < I_2 < I_3$





dynamics in is unstable

Reproduce from Euler equations:

Linearize equations of motion:

$$\omega_I(t) = \bar{\omega}_I + \delta\omega_I(t)$$

infinitesimal ... keep only first order.

special exact solution to ODE.

Euler equations to first order in δ:

$$I_1 \delta\dot{\omega}_1 = (I_2 - I_3)(\bar{\omega}_2 \delta\omega_3 + \bar{\omega}_3 \delta\omega_2) + (I_2 - I_3)\bar{\omega}_2\bar{\omega}_3 \qquad \text{etc.}$$

$(\bar{\omega}_1 = \text{const})$

Take: $\bar{\omega}_2 = \bar{\omega}_3 = 0$, but $\bar{\omega}_1 \neq 0$.

only keep linear in δ.

$$I_1 \delta\dot{\omega}_1 = 0 + \cancel{(I_2 - I_3)\delta\omega_2\delta\omega_3}$$

$$\delta\omega_1 = \text{const.} \to 0$$

$$I_2 \delta\dot{\omega}_2 = \underbrace{(I_3 - I_1)}_{>0}\bar{\omega}_1 \delta\omega_3$$

$$I_3 \delta\dot{\omega}_3 = \underbrace{(I_1 - I_2)}_{<0}\bar{\omega}_1 \delta\omega_2$$

So: $$\delta\ddot{\omega}_2 = \underbrace{\frac{(I_3 - I_1)(I_1 - I_2)}{I_2 I_3}\bar{\omega}_1^2}_{<0:\ -\Omega^2,\ \Omega \text{ real const.}} \cdot \delta\omega_2 \to \delta\omega_2 = a\cos(\Omega t) + b\sin(\Omega t)$$

not unstable: $\delta\omega_2$ small for all t.

In contrast: $\bar{\omega}_1 = \bar{\omega}_3 = 0$ but $\bar{\omega}_2 \neq 0$:

$$\delta\ddot{\omega}_1 = \underbrace{\frac{\overbrace{(I_2 - I_3)}^{<0}\overbrace{(I_1 - I_2)}^{<0}}{I_1 I_3}\bar{\omega}_2^2}_{= +\kappa^2,\ \kappa \text{ real}} \delta\omega_1 \to \delta\omega_1 = c\,e^{\kappa t} + d\,e^{-\kappa t}$$

Linear instability: rotation around 2 axis unstable.

Euler angles: one choice of coordinates on SO(3).

Euler angles

$$R_{iI} = A_{iJ''}(\phi)\, B_{J''K'}(\theta)\, A_{K'I}(\psi)$$

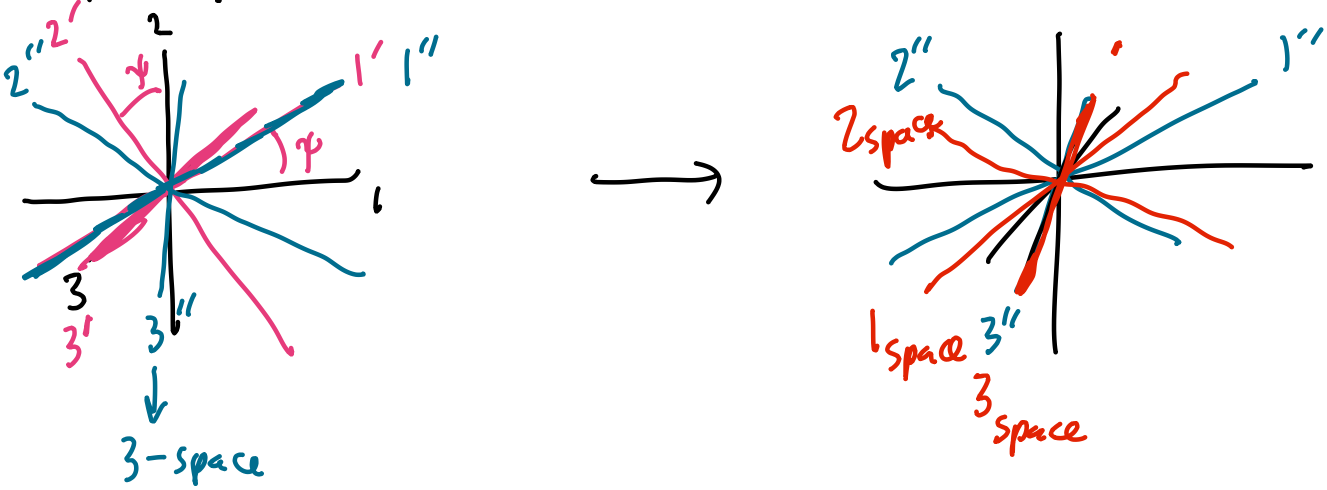
body → space rotation

specific rotation matrices

$$A(\psi) = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad B(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

rotate 3-axis $\qquad$ rotate 1-axis

If $R = A(\phi)\, B(\theta)\, A(\psi)$



Write out R:

$$R = \begin{pmatrix} \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi & -\cos\phi\sin\psi - \sin\phi\cos\theta\cos\psi & \sin\theta\sin\phi \\ \sin\phi\cos\psi + \cos\phi\cos\theta\sin\psi & -\sin\phi\sin\psi + \cos\phi\cos\theta\cos\psi & -\cos\phi\sin\theta \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{pmatrix}$$

$$\dot{R}_{iI} = R_{iJ}\,\Omega_{JI} \quad \text{(lec. 10)}$$

$$\hookrightarrow \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

$$\swarrow$$

$$\omega_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$

$$\omega_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$

$$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$$

$$L = \frac{1}{2}\dot{R}_{iI}\dot{R}_{iJ}K_{IJ} + \cdots$$

$$\hookrightarrow \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)$$