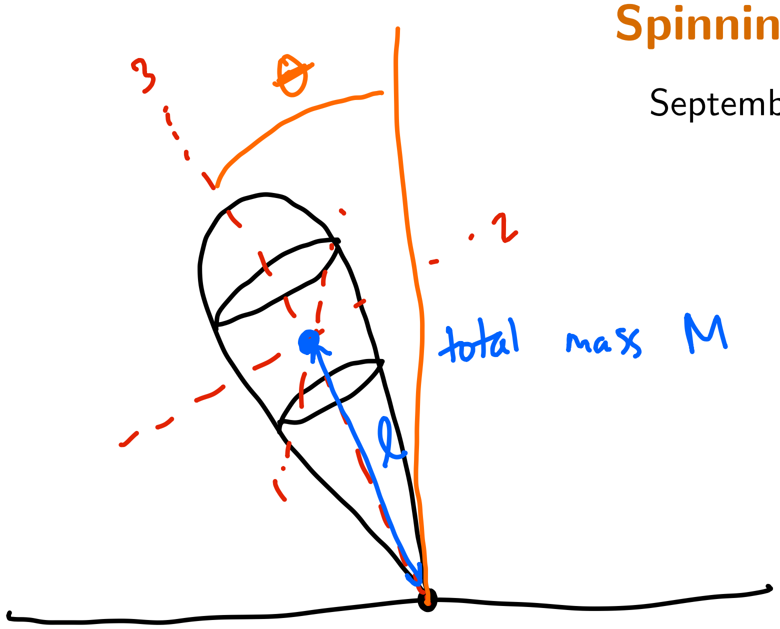
PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 12
Spinning top

September 25



$I_1 = I_2 \neq I_3$

Briefly: Euler angles

$\underbrace{(\psi, \theta, \phi)}_{}$ ...

plug in

Last time: free rigid body:

$$L = \frac{1}{2}\left(I_1 \omega_1^2 + \cancel{I_2}^{\,I_1} \omega_2^2 + I_3 \omega_3^2\right)$$

$$\to \frac{1}{2} I_1 (\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2) + \frac{1}{2} I_3 (\dot{\psi} + \dot{\phi}\cos\theta)^2 = L_0$$

Now add potential energy from gravity:

$$L = L_0 - Mg\underbrace{\ell\cos\theta}_{\text{height of center of mass}}$$

Use Noether Thm to find conserved quantities!

① $\psi \to \psi + \varepsilon$: $\quad p_\psi = \dfrac{\partial L}{\partial \dot\psi} = I_3(\dot\psi + \dot\phi\cos\theta) = I_3\omega_3$

② $\phi \to \phi + \varepsilon$: $\quad p_\phi = \dfrac{\partial L}{\partial \dot\phi} = I_1 \sin^2\theta\,\dot\phi + \cos\theta \cdot p_\psi$

③ $t \to t + \varepsilon$: $\quad \underbrace{E = \frac{1}{2}I_1(\dot\theta^2 + \sin^2\theta\,\dot\phi^2) + \frac{1}{2}I_3(\dot\psi + \dot\phi\cos\theta)^2 + Mgl\cos\theta}_{\text{plug in for } p_\phi,\, p_\psi:}$

$$E = Mgl\cos\theta + \frac{p_\psi^2}{2I_3} + \frac{1}{2}\frac{(p_\phi - p_\psi\cos\theta)^2}{I_1\sin^2\theta} + \frac{1}{2}I_1\dot\theta^2$$

Also be convenient to define $z = \cos\theta$, $\dot z = -\dot\theta\sin\theta$

$$E = Mgl \cdot z + \frac{p_\psi^2}{2I_3} + \frac{\dot z^2 + (p_\phi - p_\psi z)^2}{2I_1(1-z^2)}$$

Define:

$$a = \frac{p_\psi}{I_1}, \quad b = \frac{p_\phi}{I_1}, \quad \alpha = \frac{1}{I_1}\left(2E - \frac{p_\psi^2}{I_3}\right), \quad \beta = \frac{2Mgl}{I_1}$$

$$0 = \dot z^2 + \underbrace{(b - az)^2 - (1-z^2)(\alpha - \beta z)}_{V_{eff}(z)}$$

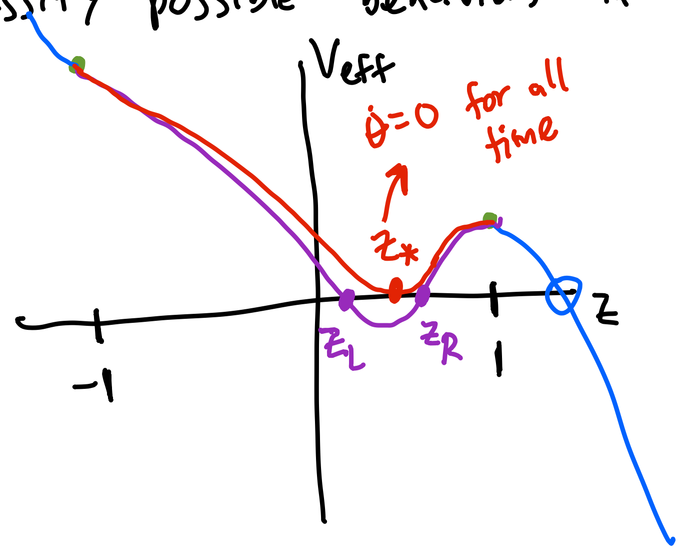
$E = T + $

(mass $m=2$)

Interpret!

Particle moving in $-1 \le z \le 1$, total energy $E = 0$.

Classify possible behaviors in $V_{eff} = \underbrace{(b-az)^2}_{\geq 0} - \underbrace{(1-z^2)}_{=0 \text{ at } z=\pm 1}(\alpha - \beta z)$ $\beta > 0$



Asymptotic behavior of $V_{eff}(z \to \pm\infty)$?

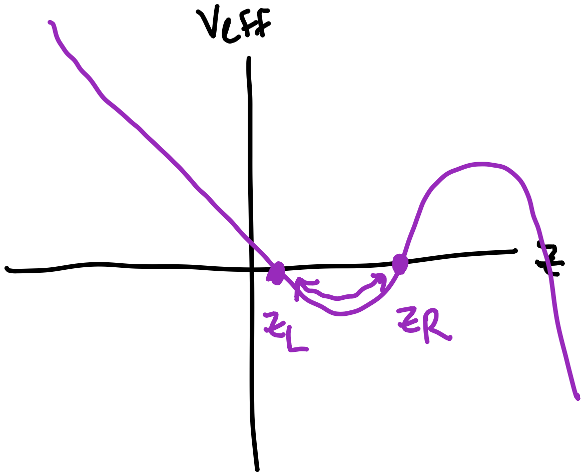
$\hookrightarrow -\beta z^3 + \dots$

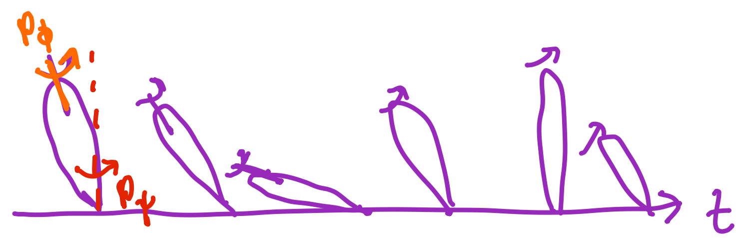
Math fact: cubic polynomial V_{eff} can have 1, 2, 3 zeroes.

fine-tuned (2) generic (3)

Physics: $E=0 \to \dot{z}^2 \geq 0$ means must be a point where $V_{eff} \leq 0$ in $-1 \leq z \leq 1$.

Generic dynamics: → precession





Simplifying assumptions (not IRL):

- neglected friction/damping...
- neglected translational motion

Consider rapidly spinning top where:

$$I_3 \omega_3^2 \gg Mg\ell$$

Usually, I_1 & I_3 similar order of magnitudes:

$$\frac{p_\psi^2}{I_3 I_1} = \frac{I_1}{I_3} a^2 \sim a^2 \gg \frac{2Mg\ell}{I_1} = \beta .$$

"Fix" b & α by: $\dot{\phi} = 0$ at $z = z_0$: $(z_0 < 1)$

- from formula for p_ϕ: $p_\phi = p_\psi z_0$ $\rightarrow$ or $b = a z_0$
- if z_0 is maximum of $z(t)$: $\dot{\theta} = 0$ also at $z = z_0$, then

$$E = \frac{1}{2} I_3 \omega_3^2 + Mg\ell z_0 \rightarrow \alpha = \beta z_0$$

Combine: $V_{eff}(z) = (z - z_0)\left[\underbrace{\overset{\text{large}}{a^2}(z - z_0)}_{\text{pos. term in } V_{eff}} + \beta\overbrace{(1 - z^2)}^{\approx \text{const} \approx 1 - z_0^2}\right]$

Estimate: angular $\Delta z \sim |\sin\theta_0| \Delta\theta \sim \frac{\beta}{a^2}$.

$$\hookrightarrow \Delta\theta \approx \frac{2Mg\ell I_1}{I_3^2 \omega_3^2 \sin\theta_0}$$

Oscillations in θ: $V_{eff} \approx$ simple harmonic oscillator w/

$$\omega_{osc} = \sqrt{\text{coeff of } z^2} = a = \frac{I_3}{I_1}\omega_3$$