PHYS 5210
Graduate Classical Mechanics
Fall 2023

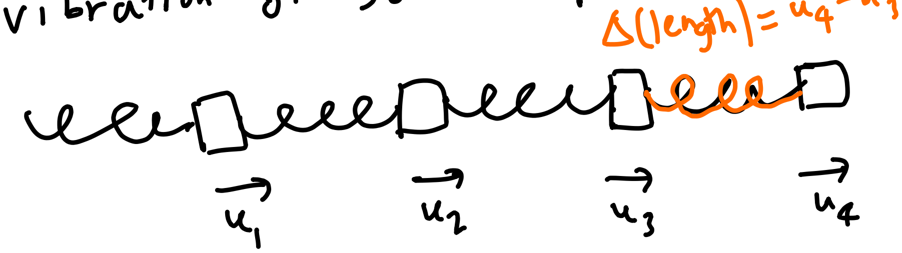
Lecture 13
Lagrangian field theory

September 27

Today: field theory describe systems of $N \gg 1$ particles w/ spatial locality.

Example: vibration of solid (phonons)

[assume identical mass m, spring constant k]



Lagrangian: $$L = \sum_{j=-\infty}^{\infty} \left[\frac{1}{2} m \dot{u}_j^2 - \frac{1}{2} k (u_{j+1} - u_j)^2 \right]$$

EOMs: $$\frac{\delta S}{\delta u_j} = 0 = -k \left[\underset{\text{term } j}{(u_j - u_{j-1})} + \underset{\text{term } j-1}{(u_j - u_{j+1})} \right] - m \ddot{u}_j$$

or $$m \ddot{u}_j = -k \underbrace{\left[2u_j - u_{j+1} - u_{j-1} \right]}_{\text{locality in space (\& time)}}.$$

Usually: only interested in long-distance physics...

Solid object: ~ 1 m vs. atomic distance $\sim 10^{-10}$ m

$\underbrace{\text{Solid object}}$ vibrating on scales $\gg 10^{-10}$ m: $\underbrace{u_j \approx u_{j+1}}$

$u_j \simeq u(j)$ [a continuous function?]

reasonable if "$\frac{du}{dj}$" $\ll u$

Then: $$2u_j - u_{j+1} - u_{j-1} \approx 2u - \left[u + \frac{\partial u}{\partial j} + \frac{1}{2}\frac{\partial^2 u}{\partial j^2} + \cdots\right] - \left[u - \frac{\partial u}{\partial j} + \frac{1}{2}\frac{\partial^2 u}{\partial j^2} - \cdots\right]$$

$$\rightarrow -\frac{\partial^2 u}{\partial j^2}$$

Convert to physical units: $a \sim 10^{-10}$ m (interatomic distance)

$j = \frac{x}{a}$, or $\frac{\partial}{\partial j} \rightarrow a\frac{\partial}{\partial x}$

Think of $u_j(t) \longrightarrow u(x,t)$ as a field, function of spatial coords (x) as well as t.

EOMs: $$m\frac{\partial^2 u}{\partial t^2} = ka^2 \frac{\partial^2 u}{\partial x^2}$$

divide by a; $\frac{m}{a} = \rho$, $\kappa = ka$

(one-dimensional mass density)

$$\rho \frac{\partial^2 u}{\partial t^2} = \kappa \frac{\partial^2 u}{\partial x^2} \quad \rightarrow \text{sound waves!}$$

Can we derive PDE directly from Lagrangian

effective field theory (EFT)?

Yes! But need to generalize Lagrangian mechanics to fields.

Idea: $$L = \sum_{j=-\infty}^{\infty} \left[\frac{a\rho}{2} \dot{u}_j^2 - \frac{\kappa}{2a} (u_{j+1} - u_j)^2 \right]$$

$$\downarrow$$

$$\approx \sum_{j=-\infty}^{\infty} a \left[\frac{\rho}{2} \left(\frac{\partial u}{\partial t}(aj, t) \right)^2 - \frac{\kappa}{2a^2} \left(a \frac{\partial u}{\partial x} + \frac{a^2}{2} \frac{\partial^2 u}{\partial x^2} + \cdots \right)^2 \right]$$

Interpret as Riemann sum for integral...

$$\sum_{j=-\infty}^{\infty} a(1 + a \cdots) \rightarrow \int_{-\infty}^{\infty} dx \, (1 + \underline{0})$$

$$L = \int_{-\infty}^{\infty} dx \, \mathcal{L} \qquad \text{where } \mathcal{L} = \frac{\rho}{2} \left(\frac{\partial u}{\partial t} \right)^2 - \frac{\kappa}{2} \left(\frac{\partial u}{\partial x} \right)^2$$

(Lagrangian density)

and action $S = \int dt \, L \rightarrow \int dt \, dx \, \mathcal{L}$

In EFT, directly write down $\mathcal{L}$ based on symmetries, locality.

E.g. solids: symmetry of

$$u(x,t) \rightarrow u(x,t) + \varepsilon.$$

Before invariant BBs... generalize Euler-Lagrange to fields:

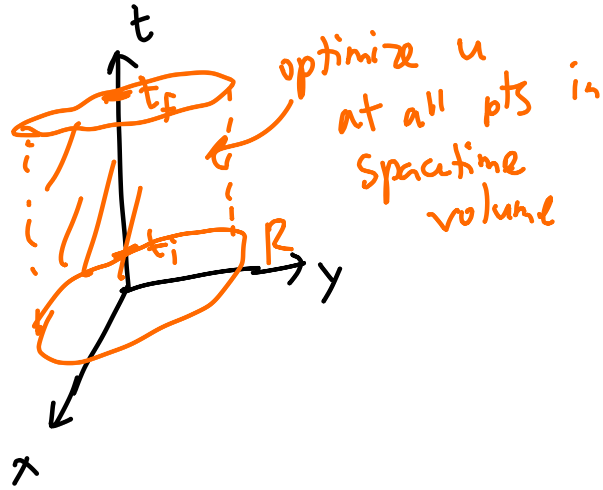
How to evaluate $\dfrac{\delta S}{\delta u(x,t)} = ?$

For ease of notation: $(x, t, \ldots) \to x^\mu$

(any add'l spatial dims)

NOT implying relativistic invariance.

$$S[u] = \int_{t_i}^{t_f} dt \int_R dV \, \mathcal{L}$$



As before, define functional derivative as:

$$\frac{d}{d\varepsilon} S[\bar{u} + \varepsilon \hat{u}]\Big|_{\varepsilon=0} = \int dt\, dV \, \hat{u} \frac{\delta S}{\delta u}\Big|_{\bar{u}}$$

POLA: $\frac{\delta S}{\delta u} = 0$ on physical "trajectories" of field.

$\partial_\mu u = \frac{\partial u}{\partial x^\mu}$

$$S[u] = \int \underbrace{d^D x}_{\text{"}\prod_\mu dx^\mu\text{"}} \, \mathcal{L}(u, \partial_\mu u, \ldots)$$

$$S[\bar{u} + \varepsilon \hat{u}] = \int d^D x \, \mathcal{L}(\bar{u} + \varepsilon \hat{u}, \partial_\mu \bar{u} + \varepsilon \partial_\mu \hat{u})$$

$$\frac{dS}{d\varepsilon} = \int d^D x \left[\hat{u} \frac{\partial \mathcal{L}}{\partial u} + \left(\sum_\mu\right) (\partial_\mu \hat{u}) \frac{\partial \mathcal{L}}{\partial(\partial_\mu u)} \right] \leftarrow \text{integrate by parts}$$

$$\hookrightarrow \underbrace{\oint d^{D-1}x \, \hat{u} \frac{\partial \mathcal{L}}{\partial(\partial_\mu u)} \hat{n}^\mu}_{= \int d^D x \, \partial_\mu \left(\hat{u} \frac{\partial \mathcal{L}}{\partial(\partial_\mu u)} \right)} - \int d^D x \, \hat{u} \, \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu u)}$$

Choose boundary conditions on $\hat{u}$ so $\hat{u} = 0$ on spacetime bdy

$$\frac{dS}{d\varepsilon} = \int d^D x \; \hat{u} \underbrace{\left[\frac{\partial \mathcal{L}}{\partial u} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu u)} \right]}_{= \frac{\delta S}{\delta u}}$$

$$\left[= \frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial t}\right)} \right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial x}\right)} \right) \right]$$

Natural generalization of $\frac{\delta S}{\delta u} = \frac{\partial L}{\partial u} - \frac{d}{dt} \frac{\partial L}{\partial\left(\frac{du}{dt}\right)}$.

$$\mathcal{L} = \frac{\rho}{2}(\partial_t u)^2 - \frac{\kappa}{2}(\partial_x u)^2$$

[shorthand: $\partial_t u = \frac{\partial u}{\partial t}$, etc.]

$$\downarrow$$

$$\frac{\delta S}{\delta u} = 0 = \cancel{\frac{\partial \mathcal{L}}{\partial u}} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial(\partial_t u)} - \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial(\partial_x u)}$$

$$= -\rho \, \partial_t^2 u + \kappa \, \partial_x^2 u$$

or: $\underline{\rho \, \partial_t^2 u = \kappa \, \partial_x^2 u}$.