PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 14
The Klein-Gordon equation

September 29

Lagrangian field theory:

$$S[\underset{\substack{\uparrow \\ \text{fields } a=1,\dots,N}}{\phi^a}(\underset{\substack{\uparrow \\ \text{spacetime coords } \mu=1,\dots,D}}{x^\mu})] = \int d^D x \; \mathcal{L}(\phi^a, \partial_\mu \phi^a, x^\mu, \dots)$$

Euler-Lagrange: $\dfrac{\delta S}{\delta \phi^a} = 0 = \underline{\dfrac{\partial \mathcal{L}}{\partial \phi^a} - \partial_\mu \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)}}$

Today: write $\mathcal{L}$ in terms of invariant BBs of symmetry

Common symmetry: spacetime translation:

$$x^\mu \to x^\mu + \underset{\substack{\uparrow \\ \text{const.}}}{\varepsilon^\mu}$$

Ensure: $\dfrac{\partial \mathcal{L}}{\partial x^\mu} = 0$, i.e. $\mathcal{L}(\phi^a, \partial_\mu \phi^a)$

Can have symmetry act on field:

shift: $\phi(x^\mu) \to \phi(x^\mu) + \underbrace{\varepsilon}_{\text{const.}}$

Invariant BBs: $\mathcal{L}(\partial_\mu \phi)$ [analogous to $x \to x+\varepsilon$ implying $L(\dot{x})$]

Suppose Lorentz invariance (relativistic):

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu$$

with $\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta_{\mu\nu} = \eta_{\rho\sigma}$

$$\begin{array}{c} \begin{array}{cccc} t & x & y & z \end{array} \\ \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} = \eta_{\mu\nu} = \eta^{\mu\nu}$$

$\partial_\mu \phi \to \Lambda_\mu{}^\nu \partial_\nu \phi$

$\Lambda_\mu{}^\nu \eta_{\nu\rho} = \eta_{\mu\sigma} \Lambda^\sigma{}_\rho$

[e.g. $\partial_x \phi \to \partial_y \phi$ via rotation]

Lorentz invariant building blocks: no dangling indices

$$\partial_\mu \phi \partial^\mu \phi = \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = -(\partial_t \phi)^2 + (\partial_x \phi)^2 + (\partial_y \phi)^2 + (\partial_z \phi)^2$$

↳ so $\mathcal{L}(\partial_\mu \phi \partial^\mu \phi, \phi)$ if Lorentz & translation symmetry

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Focus on one field $\phi(x^\mu)$.

$$S[\phi] = \int d^D x \; \mathcal{L}(\phi, \partial_\mu \phi \partial^\mu \phi)$$

if $B=0$, then EOM: $\frac{dA}{d\phi} = 0$.

dynamics ↓ $B \neq 0$.

↳ expand $\mathcal{L}$ to lowest order in derivatives

$$= \int d^D x \left[A(\phi) + B(\phi) \partial_\mu \phi \partial^\mu \phi + \cdots \right]$$

Taylor

↳ expand to lowest order in ϕ

$$= \int d^D x \left[-A_0 - A_1 \phi - \frac{A_2}{2} \phi^2 - \frac{B_0}{2} \partial_\mu \phi \partial^\mu \phi + \cdots \right]$$

Claim: ignore A_0: if $\mathcal{L} = \mathcal{L}_0 + A_0 \leftarrow$ const:

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{\partial \mathcal{L}_0}{\partial \phi} \quad \text{etc.}$$

$$S[\phi] = \underbrace{S_0[\phi]}_{\int d^Dx\, \mathcal{L}_0} + \int d^Dx\, A_0 = A_0 \times (\text{spacetime volume})$$

fixed by POLA

Claim: (relevant elsewhere) if $\mathcal{L} \to \mathcal{L} + \partial_\mu K^\mu$, physics same

$$S[\phi] \to S[\phi] + \cancel{\int d^Dx\, \partial_\mu K^\mu}$$

divergence Thm $= \oint d^{D-1}x \; n_\mu K^\mu$ — boundary term: ϕ fixed at bdy

$=$ const. for POLA.

S_0: $\mathcal{L} \approx -\frac{B_0}{2}\partial_\mu \phi \partial^\mu \phi - A_1 \phi - \frac{A_2}{2}\phi^2$

Euler-Lagrange: $0 = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = (-A_1 - A_2\phi) - \partial_\mu(B_0 \partial^\mu \phi)$

$$\frac{\partial}{\partial(\partial_\mu\phi)}\left[-\frac{B_0}{2}\partial_\alpha\phi\partial^\alpha\phi\right] = \frac{\partial}{\partial(\partial_\mu\phi)}\left[-\frac{B_0}{2}\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right]$$

$$= -\frac{B_0}{2}\eta^{\alpha\beta}\left(\delta^\mu_\alpha \partial_\beta\phi + \partial_\alpha\phi\,\delta^\mu_\beta\right) = -B_0\partial^\mu\phi$$

Look for spacetime-independent solns to EOM: $\partial_\mu\phi \to 0$ so

$$0 = -A_1 - A_2\phi \quad \text{or} \quad \phi = -\frac{A_1}{A_2}.$$

field change

$\boxed{\text{Re-define}}$ $\tilde{\phi} = \phi + \frac{A_1}{A_2}$, then $\phi = 0$ is physical "trajectory".

So $A_1 \to 0$ (w/ field re-definition).

Rescale S: change A_0, B_0

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} - \frac{m^2}{2}\tilde{\phi}^2$$

$(B_0 = 1)$ $\quad (A_2 = m^2)$

Klein-Gordon theory (relativistic "scalar" field)

Klein-Gordon equation:

$$0 = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \longrightarrow \quad m^2 \phi = \partial_\mu \partial^\mu \phi \qquad [c=1]$$

$$= (-\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2)\,\phi$$

It's often useful to calculate normal modes of an EFT.

↳ Start w/ equilibrium config: $[\phi = 0]$, and Taylor expand:

$$\phi = 0 + \delta\phi(x^\mu) \leftarrow \text{infinitesimal}$$

Evaluate EOMs at first order in perturbation:

$$m^2 \underline{\delta\phi} = \partial_\mu \partial^\mu \underline{\delta\phi}$$

If spacetime translation invariance: plug in ansatz.

$$\delta\phi(x^\mu) = \varepsilon \cdot e^{ik_\mu x^\mu} = \varepsilon e^{-i\omega t + ik_x x + \cdots}$$

ε ← const.

$$k^\mu = \begin{pmatrix} \omega \\ k_x \\ k_y \\ k_z \end{pmatrix}.$$

Using complex numbers here... intermediate steps only! Final answer for ϕ real.

Find sol'n to EOM if: $m^2 \varepsilon e^{ik_\mu x^\mu} = \partial_\mu \partial^\mu (\varepsilon e^{ik_\mu x^\mu})$

$$= -\varepsilon k_\mu k^\mu$$

$$m^2 = \omega^2 - k_x^2 - k_y^2 - k_z^2.$$

Restore speed of light $(c \neq 1)$: $\omega^2 = m^2 c^4 + c^2(k_x^2 + k_y^2 + k_z^2)$

↓

$$E^2 = (mc^2)^2 + c^2(p_x^2 + p_y^2 + p_z^2)$$

↓

Deduce K-G theory describes relativistic particles...

$\phi \rightarrow$ coherent excitations of quantum relativistic spinless particles

General solution to K-G: is a linear superposition of normal modes:

$$\omega = \pm\sqrt{k_x^2 + \dots + m^2}$$

$$\delta\phi(x^\mu) = \int dk_x dk_y dk_z \sum_{\pm} a_{\pm}(k_x, k_y, k_z)\, e^{-i\omega_{\pm}(k)t + ik_x x + \dots}$$

only integrate over k_x, k_y, k_z because $\omega^2 = m^2 + \dots$ means ω not indep.

a should be chosen so $\delta\phi(x^\mu)$ is real.

$$a_+(k_x, k_y, k_z)^* = a_+(-k_x, -k_y, -k_z)$$

[if $a_- = 0$]