# PHYS 5210
# Graduate Classical Mechanics
# Fall 2023

## Lecture 15
## Noether's Theorem in field theory

October 2

Lagrangian field theory:

$$S[\phi^a(x^\mu)] = \int d^D x \; \mathcal{L}(\phi^a, \partial_\mu \phi^a, x^\mu)$$

$\phi^a$: n fields $(a=1,\ldots,n)$

$x^\mu$: spacetime coords $(\mu=1,\ldots,D)$

S made out of invariant building blocks under symmetries.

For particles: Noether Thm: continuous symmetry $\rightarrow$ conserved quantity:

$$\frac{dQ}{dt} = 0.$$

on phys. traj.

Generalization to fields:

cont. sym $\rightarrow$ conserved current:

$$0 = \partial_\mu J^\mu = \frac{\partial J^t}{\partial t} + \frac{\partial J^x}{\partial x} + \ldots$$

$$[\partial_\mu x^\nu = \delta_\mu^\nu]$$

$$\downarrow$$

$$\partial_\mu x_\rho = \delta_\mu^\nu \eta_{\nu\rho} = \eta_{\mu\rho}$$

If we restrict to D=1: $0 = \partial_\mu J^\mu = \partial_t Q$

Derivation of following is similar to lecture 3:

Define continuous symmetry: infinitesimal transformation...

- $\mathcal{L}(x^\mu, \phi^a) = \mathcal{L}(\tilde{x}^\mu, \tilde{\phi}^a) + \varepsilon\, \partial_\mu K^\mu$ [$\varepsilon$ infinitesimal]
- $\tilde{x}^\mu = x^\mu + \varepsilon X^\mu$
- $\tilde{\phi}^a = \phi^a(x^\mu) + \varepsilon\, \varphi^a(x^\mu) = \phi^a(\tilde{x}^\mu) + \varepsilon\left[\varphi^a(\tilde{x}^\mu) - X^\nu \partial_\nu \varphi^a(\tilde{x})\right] + \ldots$

where $S[\tilde{\phi}] = S[\phi]$. (action is invariant)

$S$ invariant if:

$$0 = \mathcal{L}\,\partial_\mu X^\mu + \partial_\mu K^\mu + X^\mu \frac{\partial \mathcal{L}}{\partial x^\mu} + \varphi^a \frac{\partial \mathcal{L}}{\partial \phi^a} + \left(\partial_\nu \varphi^a - \partial_\nu X^\mu \partial_\mu \varphi^a\right)\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi^a)}$$

If satisfied $\rightarrow$ continuous symmetry.

Noether's Thm: on solutions to EOM, $\partial_\mu J^\mu = 0$ where

$$J^\mu = X^\mu \mathcal{L} + K^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)}\left(\varphi^a - X^\nu \partial_\nu \varphi^a\right)$$

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$\partial_\mu J^\mu = 0$ implies there's $Q$ with $\frac{dQ}{dt}$:

$$Q = \int \underbrace{dx\,dy\,dz}_{\text{NOT time}}\; J^t \quad \text{density of "}Q\text{"}$$

$$\frac{dQ}{dt} = \int dx \ldots \frac{\partial J^t}{\partial t} = \underbrace{\int dx \cdots \left(-\frac{\partial J^x}{\partial x} - \cdots\right) = 0}_{\text{if spatial bdy doesn't add/remove charge (almost always)}}$$

Example 1: massless Klein-Gordon theory

$$S[\phi] = -\frac{1}{2}\int d^Dx \; \partial_\mu \phi \partial^\mu \phi \qquad \underline{(m=0)}$$

Shift symmetry: $\phi \to \phi + \varepsilon$.

$$\varphi = 1, \quad X^\mu = 0, \quad K^\mu = 0$$

Conserved current: $J^{\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)} = -\partial^\mu \phi$

raise

lower

$$\partial_\mu J^\mu = -\partial_\mu \partial^\mu \phi = 0$$

$$\hookrightarrow \frac{\delta S}{\delta \phi} = \cancel{\frac{\partial \mathcal{L}}{\partial \phi}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0.$$

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Example 2: spacetime translation symmetry (almost always!)

$D$ symmetries $\leftarrow X^\mu = \text{const.} \leftarrow \left[\frac{\partial \mathcal{L}}{\partial x^\mu} = 0\right]$

① $X^1 = 1$, $X^2 = 0$, $\vdots$ ② $X^1 = 0$, $X^2 = 0$, $\vdots$ … Ⓓ $X^1 = 0$, $\vdots$, $X^D = 1$

Package together: $X^\mu = \overset{\text{const.}}{\varepsilon^\mu} \rightarrow X^\mu = \delta^\mu_\nu \, \varepsilon^\nu$

$\nu$ = choice of symmetry

$\mu$ index = label of spatial coords

Symmetry holds for any $\varepsilon^\nu \rightarrow$ $X^\mu = \delta^\mu_\nu \; (\times \varepsilon^\nu)$

$$K^\mu = 0$$

$$\varphi^a = 0.$$

Noether Thm: $J^\mu(\varepsilon) = \underbrace{J^\mu{}_\nu}\, \varepsilon^\nu$

$$J^\mu{}_\nu = \delta^\mu_\nu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta^\rho_\nu \partial_\rho \phi = \delta^\mu_\nu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi$$

$J^\mu{}_\nu$ is conserved $[\partial_\mu J^\mu{}_\nu = 0]$ for $x^\nu$ translation.

In relativistic theories:

pick one "word" 1

pick one "word" 2

$$J^\mu{}_\nu \eta^{\nu\rho} \longmapsto T^{\mu\rho} =$$

energy-momentum / stress / stress-energy

current / tensor / tensor

all combos OK.

$$\partial_\mu T^{\mu\rho} = 0, \qquad T^{\mu\rho} = \eta^{\mu\rho}\mathcal{L} - \partial^\rho \phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$$

Conservation of <u>energy</u> $(X^t = 1)$: $(\mu = t)$

$$E = \int \underbrace{d^3x}_{\text{space coords only}} \; T^{t\cancel{t}}$$

density of conserved

time-component $\nu$ (energy cons)

Klein-Gordon: $\mathcal{L} = -\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2}\phi^2 = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_x \phi)^2 \ldots$

$$T^{tt} = -\mathcal{L} - (\overset{-1}{\cancel{\eta^{tt}}}\partial_t \phi) \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)}$$

$$= \underbrace{\frac{1}{2}(\partial_t \phi)^2}_{\text{"kinetic" energy}} + \underbrace{\frac{1}{2}(\partial_x \phi)^2}_{\text{"strain" potential energy}} + \ldots + \underbrace{\frac{1}{2}m^2\phi^2}_{\text{"harmonic osc. potential"}}$$