PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 16
Electromagnetism

October 4

Last time: Noether's Thm: continuous symmetry

$$\mathcal{L}(x) \mapsto \mathcal{L}(x) - \varepsilon \partial_\mu K^\mu, \quad \phi^a \to \phi^a + \varepsilon \varphi^a, \quad x^\mu \to x^\mu + \varepsilon X^\mu$$

↳ conserved current J^μ: $[\partial_\mu J^\mu = 0]$

$$J^\mu = X^\mu \mathcal{L} + K^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)}(\varphi^a - X^\nu \partial_\nu \varphi^a)$$

Usually: spacetime translation symmetry: $\frac{\partial \mathcal{L}}{\partial x^\mu} = 0.$

↳ conserved stress tensor

$T^{\mu\nu}$ obeying $\partial_\mu T^{\mu\nu} = 0$

μ'th component of conserved current

x^ν-translation symmetry

Relativistic: $$T^{\mu\nu} = \eta^{\mu\nu}\mathcal{L} - \partial^\nu \phi^a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)}$$

Claim: Lorentz invariance $\rightarrow$ $T^{\mu\nu} = T^{\nu\mu}$

(lec 6) $x^\mu \rightarrow x^\mu + \underbrace{\varepsilon^\mu{}_\nu x^\nu}_{\text{(infinitesimal)}}$ with $\varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}$

Use Noether's Thm: $X^\mu = \underline{\varepsilon^\mu{}_\nu} x^\nu$ so

$$\tilde{J}^\mu(\varepsilon) = X^\nu T^\mu{}_\nu = \varepsilon^\nu{}_\rho x^\rho T^\mu{}_\nu = \varepsilon_{\nu\rho} x^\rho T^{\mu\nu}$$

$\tilde{J}^\mu$ conserved:

$$\partial_\mu \tilde{J}^\mu = 0 = \partial_\mu(\varepsilon_{\nu\rho} x^\rho T^{\mu\nu}) = \varepsilon_{\nu\rho}\delta^\rho_\mu T^{\mu\nu} + \varepsilon_{\nu\rho} x^\rho \overset{0}{\cancel{\partial_\mu T^{\mu\nu}}}$$

$$= \underline{\varepsilon_{\nu\mu} T^{\mu\nu}}$$

e.g. Set $\varepsilon_{xy} = -\varepsilon_{yx} = 1$: $\hookrightarrow$ $0 = T^{xy} - T^{yx}$
rest $= 0$

So: $T^{\nu\mu} = T^{\mu\nu}$.

Today: EFT for electromagnetism.

Recall: E&M has physical fields $\vec{E}, \vec{B}$:

constraints
$\nabla \cdot \vec{B} = 0$
$\nabla \times \vec{B} = -\frac{\partial E}{\partial t}$

$$\vec{E} = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \nabla \times \vec{A}$$

lec 7: $\vec{E}$ & $\vec{B}$ in terms of gauge field $A_\mu = (-\varphi, \vec{A})$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \longrightarrow \begin{array}{cc} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} & \begin{array}{c} t \\ x \\ y \\ z \end{array} \end{array}$$

(in units where $c=1$)

Goal: Lagrangian that leads to Maxwell equations (in free space)

Note: $\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0.$ $\nabla\cdot B = 0$ $\rho\mu\nu = xyz$

$= (\partial_\rho\partial_\mu - \partial_\mu\partial_\rho)A_\nu + \cdots = 0.$ $\nabla\times B = -\frac{\partial E}{\partial t}$ $\rho = t.$

Constraint: A_μ is not uniquely defined due to "gauge symmetry"

$A_\mu \to A_\mu + \partial_\mu \lambda(x)$ for any $\lambda(x)$

Best: think of A_μ and $A_\mu + \partial_\mu\lambda$ as "same points" in configuration sp

Simplest gauge-invariant building block: $F_{\mu\nu}$.

+ translation / Lorentz: $F_{\mu\nu}F^{\mu\nu}$

$$\hookrightarrow \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \underbrace{b(F_{\mu\nu}F^{\mu\nu})^2 + a\,\partial_\rho F_{\mu\nu}\partial^\rho F^{\mu\nu} + \cdots}_{\text{irrelevant (subleading): } 4\ \partial}$$

Do we get Maxwell's equations?

$$\frac{\delta S}{\delta A_\alpha} = 0 = -\partial_\beta \frac{\partial \mathcal{L}}{\partial(\partial_\beta A_\alpha)} = \frac{1}{4}\partial_\beta\left[\frac{\partial}{\partial(\partial_\beta A_\alpha)}\left(\eta^{\mu\rho}\eta^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}\right)\right]$$

$$\frac{\partial F_{\mu\nu}}{\partial(\partial_\beta A_\alpha)} = \delta^\mu_\beta\delta^\nu_\alpha - \delta^\nu_\beta\delta^\mu_\alpha$$

$$0 = \frac{1}{4}\cdot 2\,\eta^{\mu\rho}\eta^{\nu\sigma}\partial_\beta\left(F_{\rho\sigma}(\delta^\mu_\beta\delta^\nu_\alpha - \delta^\nu_\beta\delta^\mu_\alpha)\right)$$

$$= \frac{1}{2}\partial_\beta\left(F^{\mu\nu}(\cdots)\right) = \frac{1}{2}\partial_\beta(F^{\beta\alpha} - F^{\alpha\beta}) = \partial_\beta F^{\beta\alpha}$$

$\alpha = t$: $\quad 0 = \partial_i F^{it} = \partial_i(-E^i)$ $\quad$ [Gauss' Law]

(ij = space only)

$\alpha = j$: $\quad 0 = \partial_t F^{tj} + \partial_i F^{ij} = +\partial_t E_j + \partial_i(\varepsilon_{ijk} B_k)$

$= [\partial_t \vec{E} - \nabla \times \vec{B}]_j$ $\quad$ [Ampere-Maxwell]

Normal modes of EFT: $\quad \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

linearize: $\quad A_\mu = 0 + \delta A_\mu \to A_\mu.$

$$\partial_\mu F^{\mu\nu} = 0 = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

Spacetime trans symmetry: $\quad A^\mu = \underset{\text{const.}}{a^\mu} e^{ik_\alpha x^\alpha} = a^\mu e^{-i\omega t + i\vec{k}\cdot\vec{x}}$

on plane waves: $\quad \partial_\mu \to i k_\mu \qquad \partial^\mu \to ik^\mu$

$$k^\mu = \begin{pmatrix} \omega \\ k_x \\ k_y \\ k_z \end{pmatrix} \longrightarrow \begin{pmatrix} \omega \\ k \\ 0 \\ 0 \end{pmatrix} \quad \text{[align x-axis w/ } \vec{k}\text{]}.$$

$$0 = -k_\mu k^\mu a^\nu + k^\nu k_\mu a^\mu$$

Take $\nu = y, z$: $\quad 0 = \underbrace{(-\omega^2 + k^2)}_{=0:\ \omega = \pm k} a^{y,z}$ $\quad$ [$\omega = \pm ck$] $\quad$ photon

Take $\nu = t, x$: $\quad -(\omega^2 - k^2) a^t = \omega(-\omega a^t + k a^x)$

$-(\omega^2 - k^2) a^x = k(-\omega a^t + k a^x)$

$$\hookrightarrow \frac{a^t}{a^x} = \frac{\omega}{k}$$

So: $a^t = i\omega \cdot \text{const}$, $a^x = ik \cdot \text{const}.$

$$\hookrightarrow \underline{a^\mu = \partial^\mu e^{ik_\alpha x^\alpha}}$$

gauge redundancy
unphysical.

Summarize: normal modes of EFT of electromagnetism
= 2 photons