PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 17
Coupling electromagnetism to matter

October 6

How E & M couples to matter:

$$\nabla\cdot\vec{E}=\rho \qquad \nabla\times\vec{B}=\vec{J}+\partial_t\vec{E} \quad \rightsquigarrow \quad \partial_\mu F^{\mu\nu}=\overset{(\rho,\vec{J})}{J^\nu}$$

Goal: derive this from EFT.

What kinds of FTs/matter couple to E&M?

$$\partial_\nu\partial_\mu \underbrace{F^{\mu\nu}}_{\text{antisym.}} = \partial_\nu J^\nu = 0 \qquad \text{so } \underline{J^\nu \text{ conserved}}.$$

symmetry: U(1) symmetry
phase rotation sym.

Claim: E&M couples to theories w/ U(1) symmetry

Minimal example:

$$\mathcal{L}=-\frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1-\frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2-\frac{1}{2}m^2(\phi_1^2+\phi_2^2)$$

$$U(1):\quad \begin{pmatrix}\tilde{\phi}_1\\ \tilde{\phi}_2\end{pmatrix}=\begin{pmatrix}\cos\lambda & \sin\lambda\\ -\sin\lambda & \cos\lambda\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}:\quad \mathcal{L}(\phi_1,\phi_2)=\mathcal{L}(\tilde{\phi}_1,\tilde{\phi}_2)$$

More elegant: $\phi = \phi_1 + i\phi_2$ $\quad \bar{\phi} = \phi_1 - i\phi_2$

complex-valued, ϕ and $\bar{\phi}$ have same info

complex conjugates

change of basis!

U(1): $\phi \to \phi e^{i\lambda}$ $\quad \bar{\phi} \to \bar{\phi} e^{-i\lambda}$

by convention rescale $\mathcal{L}$.

Klein-Gordon for complex fields: $\mathcal{L} = -\cancel{\tfrac{1}{2}}\partial_\mu \phi \partial^\mu \bar{\phi} - \cancel{\tfrac{1}{2}} m^2 \phi\bar{\phi}$

U(1) symmetry manifest:

$$\phi\bar{\phi} \to (\phi e^{i\lambda})(\bar{\phi} e^{-i\lambda}) = \phi\bar{\phi}\, e^{i\lambda - i\lambda} = \phi\bar{\phi}$$

<u>Conserved current</u> associated w/ U(1) symmetry:

$$\phi \to e^{i\lambda}\phi \to \underbrace{(1+i\lambda)\phi}_{\varphi = i\phi} \qquad \bar{\phi} \to e^{-i\lambda}\bar{\phi} \approx (1-i\lambda)\bar{\phi}$$

$$\bar{\varphi} = -i\bar{\phi}$$

Noether Thm (lec 15): $J^\mu = \varphi \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} + \bar{\varphi}\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\phi})}$

$$= -i\left(\phi \partial^\mu \bar{\phi} - \bar{\phi}\partial^\mu \phi\right)$$

So far: global U(1) symmetry $\quad [\lambda \text{ const.}]$

Couple theory to <u>dynamical</u> E&M promote U(1) into "gauge symmetry"

$$\phi(x) \to e^{i\lambda(x)}\phi(x) \quad \text{and} \quad \bar{\phi}(x) \to e^{-i\lambda(x)}\bar{\phi}(x).$$

Naive invariant building blocks?

$$\phi\bar{\phi} \to \cancel{e^{i\lambda(x) - i\lambda(x)}}\,\phi\bar{\phi} \quad ✓ \qquad \partial_\mu(\phi\bar{\phi})\partial^\mu(\phi\bar{\phi})$$

etc...

$$\partial_\mu \phi \partial^\mu \bar{\phi} \to \partial_\mu(\phi e^{i\lambda}) \partial^\mu (e^{-i\lambda}\bar{\phi})$$
$$= \partial_\mu \phi \partial^\mu \bar{\phi} + \underbrace{\phi \, i \partial_\mu \lambda \, \partial^\mu \bar{\phi} + \cdots}$$

$\hookrightarrow$ $\partial_\mu \phi \partial^\mu \bar{\phi}$ can't be invariant BB :(

Idea: we also have E&M: $A_\mu \to A_\mu + \partial_\mu \lambda$ same λ

Define covariant derivatives

$$D_\mu \phi = \partial_\mu \phi - i A_\mu \phi \qquad D_\mu \bar{\phi} = \partial_\mu \bar{\phi} + i A_\mu \bar{\phi}$$

Under gauge transform $\lambda(x)$:

$$D_\mu \phi \to \partial_\mu(\phi e^{i\lambda}) - i(A_\mu + \partial_\mu \lambda)\phi e^{i\lambda}$$
$$= e^{i\lambda}\partial_\mu \phi + \cancel{\phi e^{i\lambda} i \partial_\mu \lambda} - i A_\mu \phi e^{i\lambda} - \cancel{\phi e^{i\lambda} i \partial_\mu \lambda}$$
$$= e^{i\lambda} D_\mu \phi$$

Similarly: $D_\mu \bar{\phi} \to e^{-i\lambda} D_\mu \bar{\phi}$.

To recap: invariant BBs under $U(1)$ gauge symmetry:

$$\phi \bar{\phi}, \quad D_\mu \phi D^\mu \bar{\phi}, \quad \underbrace{F_{\mu\nu} F^{\mu\nu}}_{\text{lec. 16}} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

Minimal EFT for E&M + matter:

$$\mathcal{L} = -D_\mu \phi D^\mu \bar{\phi} - m^2 \bar{\phi}\phi - \underline{\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}} + \cdots \quad (F_{\mu\nu}F^{\mu\nu})^2 + \cdots$$

Check: do we recover Maxwell?

$$\frac{\delta S}{\delta A_\alpha} = \underbrace{\partial_\beta F^{\beta\alpha}}_{\text{lec. 16}} + \underbrace{\frac{\partial}{\partial A_\alpha}\left(D_\mu \phi D^\mu \bar{\phi}\right)}$$

$$\cdots + D^\mu \bar{\phi} \frac{\partial D_\mu \phi}{\partial A_\alpha} = D^\mu \bar{\phi} \frac{\partial(\partial_\mu \phi - i A_\mu \phi)}{\partial A_\alpha}$$

$$= D^\mu \bar{\phi} \cdot (-i\phi \delta^\alpha_\mu) + \cdots$$

$$= -i(D^\mu \bar{\phi})\phi + i\bar{\phi} D^\mu \phi$$

$$= -J^\mu$$

definition of J^μ with gauge field A

Altogether: $\underline{\partial_\beta F^{\beta\alpha} = J^\alpha}$ Maxwell!

Here, J^μ similar to Noether current for global U(1), w/ $\partial^\mu \to D^\mu$.

Similarly:

$$\frac{\delta S}{\delta \bar{\phi}} = \frac{\partial \mathcal{L}}{\partial \bar{\phi}} - \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu \bar{\phi})}$$

$$-m^2\phi - D_\mu \phi \underbrace{\frac{\partial}{\partial \bar{\phi}}(D^\mu \bar{\phi})}_{+iA^\mu} - \partial_\nu \underbrace{\frac{\partial(D^\mu \bar{\phi})}{\partial(\partial_\nu \bar{\phi})}}_{-\eta^{\mu\nu}} D_\mu \phi = 0$$

or $\quad 0 = -m^2\phi - iA^\mu D_\mu \phi + \partial^\mu D_\mu \phi$

$$m^2\phi = (\partial^\mu - iA^\mu) D_\mu \phi = D^\mu D_\mu \phi$$

define as