PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 18
Effective field theory of solids

October 9

Goal: EFT for solids
1. What are fields?
2. Symmetries?
3. Invariant building blocks
4. Build $\mathcal{L}$ (stability!)

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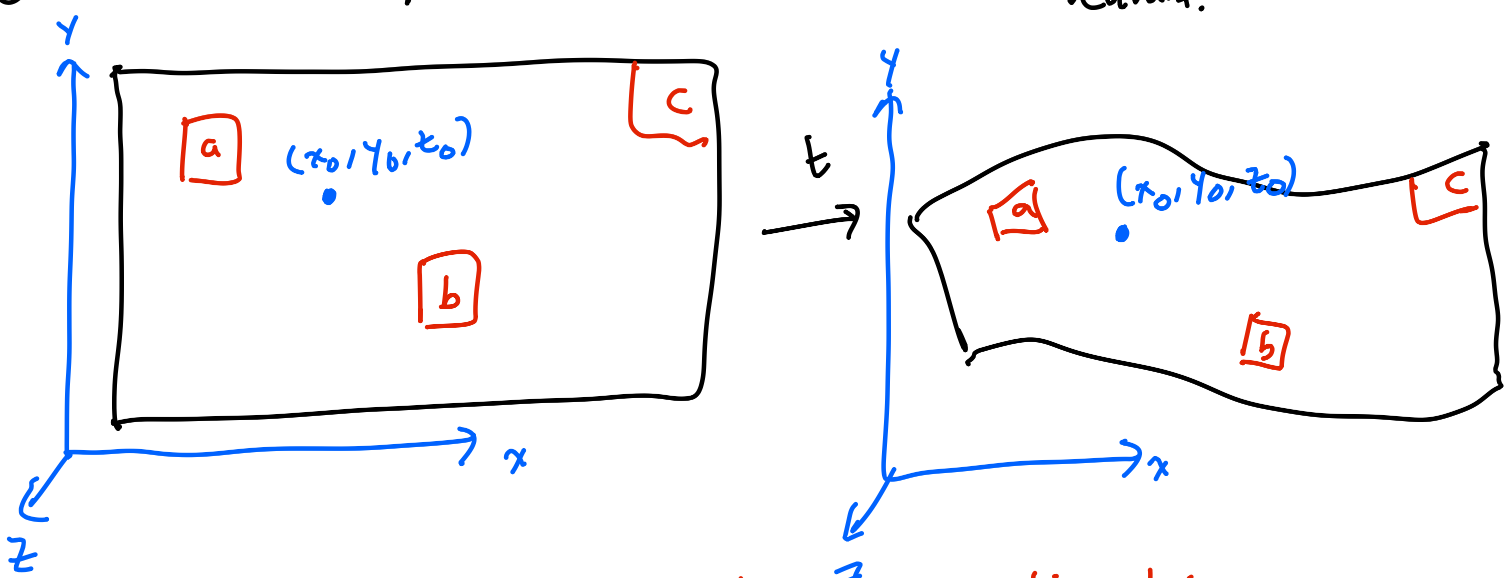
1. Describe dynamics of continuous medium.



What chunk is at (x_0, y_0, z_0) at each time t:

$\sigma \longrightarrow$ field: $\sigma(x, t)$

Eulerian frame

More precisely: if solid is in equilibrium

$$\underbrace{\sigma = x}_{\longrightarrow} \qquad \sigma^I = \delta^I_i x_i$$

Fields are $(\sigma^1, \sigma^2, \sigma^3)$ $(I = 1, 2, 3)$.

Analogy to rigid body: σ^I: x_i $\longrightarrow$ $\sigma^I(x_i)$

"space frame" "body frame"

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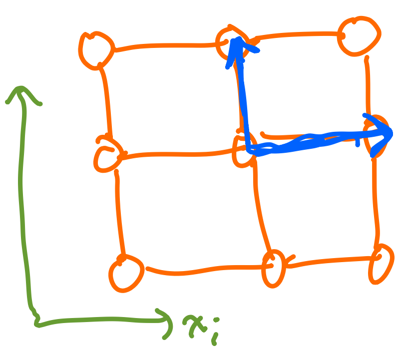
② Symmetries.

- time-translation : $t \to t + \varepsilon$.
- space-trans : $x_i \to x_i + \varepsilon_i$
- also translation: $\sigma^I \to \sigma^I + \varepsilon^I$

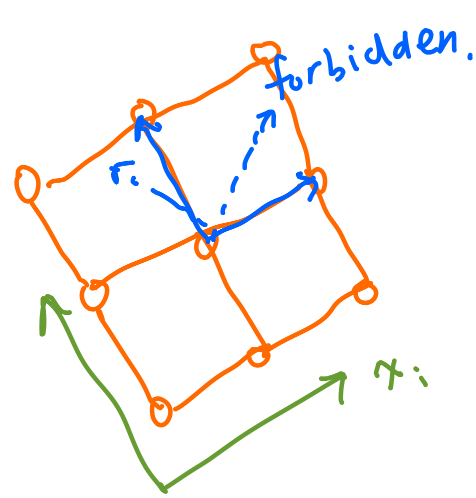
if equilibrium is $\sigma^I = \delta^I_i x_i$,
this config is only invariant if $\sigma^I \to \sigma^I + \varepsilon^I$
$x_i \to x_i + \delta^I_i \varepsilon^I$

→ spontaneous symmetry breaking of translations.

- space frame rot: $x_i \to Q_{ij} x_j$ ↙ orthogonal $(\in SO(3))$
- in class: isotropic solid : $\sigma^I \to R^{IJ} \sigma^J$



cubic lattice



SSB of rotation also.

x_i symmetries $\to$ space frame (always!)

σ^I symmetries $\to$ body frame, symmetry of crystal (explicitly broken!)

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③ Invariant BBs:

- x_i & t - trans: $\mathcal{L}$ has no explicit dep on x or t.
  $\to \mathcal{L}(\sigma^I, \partial_t \sigma^I, \partial_i \sigma^I \ldots)$
- σ^I trans: $\mathcal{L}(\partial_t \sigma^I, \partial_i \sigma^I, \ldots)$
- x_i -rot: ij indices contracted.
  [ε_{ijk} not allowed if $x \to -x$ sym.]
- in class: isotropic solid
  ↳ also contract IJ indices.

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④ Need $\mathcal{L}$ to have stable equilibria on:

$\sigma^I \to \delta^I_i x_i + \delta^I_i \varepsilon_i$ $\quad \sigma^I = \delta^I_i x_i \quad \sigma^I = \delta^I_i (Q_{ij} x_j)$

(SSB) $\quad x_i \to x_i + \varepsilon_i \quad$ rotations $x \to Qx \quad$ Q orthogonal

How to find invariant BBs that vanish in equilibrium?

- $\partial_t \sigma^I \partial_t \sigma^I$ OK
- "$\partial_i \sigma^I \partial_i \sigma^J - \delta^{IJ}$" [if IJ contracted later!]
  ↳ $\partial_i(\delta^I_j x_j)\, \partial_i(\delta^J_k x_k) - \delta^{IJ}$
  $= \underbrace{\delta^I_j \delta_{ij}}_{\delta^I_i} \underbrace{\delta^J_k \delta_{ik}}_{\delta^J_i} - \delta^{IJ} = \delta^I_i \delta^J_i - \delta^{IJ} = 0$ ✓.

$Q_{jl} x_l \quad Q_{ji}$

Use these invariant BBs:

$$\mathcal{L} = \underbrace{\frac{\rho}{2}\partial_t\sigma^I\partial_t\sigma^I}_{\rho \,=\, \text{mass density}} - \frac{1}{8}\lambda^{\underline{IJ}\,\underline{KL}}\left(\partial_i\sigma^I\partial_i\sigma^J - \delta^{IJ}\right)\left(\partial_j\sigma^K\partial_j\sigma^L - \delta^{KL}\right)$$

Choose λ^{IJKL} so $\frac{1}{8}\lambda\cdots \geq 0$, then stability.

[Note: under body frame: $\sigma^I \to R^{II'}\sigma^{I'}$
$\lambda^{IJKL} \to R^{II'}R^{JJ'}R^{KK'}R^{LL'}\lambda^{I'J'K'L'}$]

For isotropic solid: want $\lambda^{IJKL} \to \lambda^{IJKL}$ under R-rot.

So build λ^{IJKL} only out of δ's.

$$\lambda^{\underline{IJ}\,KL}[\text{symmetric}]^{IJ} \quad : \quad \lambda^{IJKL} = \lambda^{JIKL}$$

$$\lambda^{\underline{IJ}\,\underline{KL}}(\#)^{IJ}(\#)^{KL} = \lambda^{\underline{KL}\,\underline{IJ}}(\#)^{KL}(\#)^{IJ} \quad : \quad \lambda^{IJKL} = \lambda^{KLIJ}$$

λ^{IJKL} = 4^{th} rank tensor (4 indices)

So:

$$\lambda^{IJKL} = K\delta^{IJ}\delta^{KL} + \tilde{\mu}\left(\delta^{IK}\delta^{JL} + \delta^{IL}\delta^{JK}\right)$$