PHYS 5210

Graduate Classical Mechanics

Fall 2023

Lecture 19

Elastic solids

October 11

Degrees of freedom in solid:

$\sigma^I(x_i,t)$ = equilibrium location of piece of solid is now at x_i at time t. $I = 1,2,3$.

$$\mathcal{L} = \frac{\rho}{2}\partial_t\sigma^I\partial_t\sigma^I - \frac{1}{8}\lambda^{IJKL}\left(\partial_i\sigma^I\partial_i\sigma^J - \delta^{IJ}\right)\left(\partial_j\sigma^K\partial_j\sigma^L - \delta^{KL}\right)$$

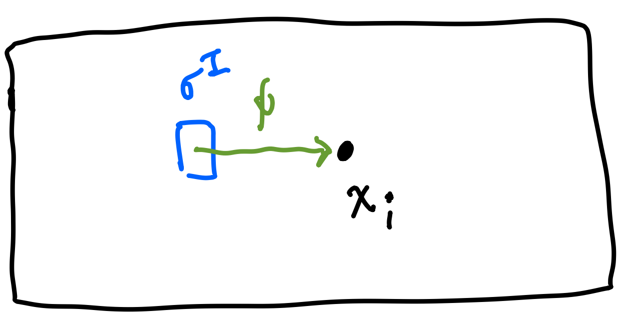
$$\lambda^{IJKL} = \tilde{K}\delta^{IJ}\delta^{KL} + \mu\left(\delta^{IK}\delta^{JL} + \delta^{IL}\delta^{JK}\right)$$

for isotropic solid, λ can only contain δ's $\to$ HW7 relaxes.

Today: behavior of solids close to equilibrium.

$$\sigma^I = \delta^I_i(x_i - \phi_i)$$

infinitesimally small, only keep $O(\phi)$ terms in EOMs.



Expand $\mathcal{L}$ to quadratic order in ϕ:

- $\partial_t \sigma^I \quad = \quad \delta^I_i(-\partial_t \phi_i)$
- $\partial_i \sigma^I \partial_i \sigma^J - \delta^{IJ} = \partial_i\left[\delta^I_m(x_m - \phi_m)\right]\partial_i\left[\delta^J_n(x_n-\phi_n)\right] - \delta^{IJ}$

$$= \delta^I_m \delta^J_n\left(\delta_{im}\delta_{in} - \delta_{im}\partial_i\phi_n - \delta_{in}\partial_i\phi_m + \cancel{\partial_i\phi_m\partial_i\phi_n}\right) - \delta^{IJ}$$

$$= \cancel{\delta^{IJ}} - \cancel{\delta^{IJ}} - \delta^I_m\delta^J_n\underbrace{(\partial_m\phi_n + \partial_n\phi_m)} + \text{quadratic } \phi$$

Strain tensor: $u_{mn} = \dfrac{\partial_m\phi_n + \partial_n\phi_m}{2}$

$$\mathcal{L} \approx \frac{\rho}{2}\partial_t\phi_i\partial_t\phi_i - \frac{1}{8}\lambda^{IJKL}\left[\delta^I_m\delta^J_n(-\partial_m\phi_n - \partial_n\phi_m)\right]\left[\delta^K_p\delta^L_q(-\partial_p\phi_q - \partial_q\phi_p)\right]$$

$$= \frac{\rho}{2}\partial_t\phi_i\partial_t\phi_i - \frac{1}{8}\lambda_{mnpq}(\partial_m\phi_n + \partial_n\phi_m)(\partial_q\phi_p + \partial_p\phi_q)$$

(Vestigial) symmetries?

- x_i & t translation $\quad$ [$x_i \to -x_i$ $\quad t \to -t$ also!]
- $\phi_i \to \phi_i + \varepsilon_i$ $\quad$ shift symmetry $\quad$ [combo of x & σ shift]

  global displacement of solid OK.

- x_i rotation symmetry:

  ① if $\lambda_{mnpq} = \tilde{K}\delta_{mn}\delta_{pq} + \cdots$ then $x_i \to Q_{ij}x_j$ (orthogonal)

  ② $\mathcal{L} = 0$ if $\phi_i = \varepsilon_{ij}x_j$ $\quad$ $\varepsilon_{ij} = -\varepsilon_{ji}$ antisym.

$\partial_i\phi_j + \partial_j\phi_i = \varepsilon_{ji} + \varepsilon_{ij} = 0.$

$\hookrightarrow$ σ^I rotation symmetry $\rightsquigarrow$ symmetries of λ_{ijkl}

- $\phi_i \to \varepsilon_{ij}x_j$ encodes space frame

  $x_i \to Q_{ij}x_j$ rotation of $\mathcal{L}(\sigma^I)$.

Physical predictions of theory:

- normal modes (lec. 20)
- stress on solids (time-independent)

Calculate stress tensor from Noether's Thm.

lec 15: stress tensor from x_i translation... in $\mathcal{L}(\sigma, x)$ [not $\mathcal{L}(\phi, x)$].

$$T_{ij} = \mathcal{L}\delta_{ij} - \partial_j\sigma^I \frac{\partial\mathcal{L}}{\partial(\partial_i\sigma^I)}$$

$$= \underbrace{\mathcal{L}}_{\phi^2}\delta_{ij} - \underbrace{\partial_j\sigma^I}_{\approx\delta^I_j\ (\phi^0)}\left[-\frac{1}{4}(\lambda^{IJKL}+\lambda^{JIKL})\underbrace{\partial_i\sigma^J}_{\phi^0}\underbrace{(\partial_j\sigma^K\partial_j\sigma^L - \delta^{KL})}_{\phi+\phi^2}\right]$$

Lowest order in ϕ? Linear in ϕ...

$$T_{ij} \approx \delta^I_j \frac{1}{2}\lambda^{IJKL}\delta^J_i(-\delta^K_m\delta^L_n)(\partial_m\phi_n + \partial_n\phi_m)$$

$$= -\frac{1}{2}\lambda_{ijkl}(\partial_k\phi_l + \partial_l\phi_k) = -\lambda_{ijkl}u_{kl}$$

Theory of elastic solids:

stress (T) linearly prop to strain (u)

When is equilibrium stable? "$\lambda_{ijkl} \geq 0$"

[analogue to Hooke's Law: $F = -kx$, $k>0$].

Lamé constants / elastic moduli

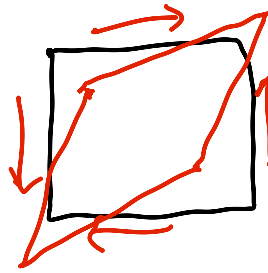
Isotropic solid: $\lambda_{ijkl} = \tilde{K}\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$

Potential energy: $\sim \frac{1}{2}\lambda_{ijkl}u_{ij}u_{kl} \geq 0$ [term in $\mathcal{L}$]

Check this positivity:

① $u_{xy} = u_{yx} \neq 0$

$$u = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

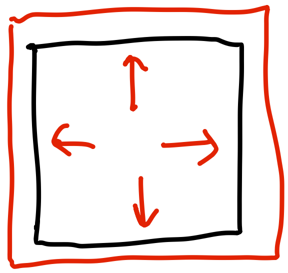


$\leftarrow \partial_x \phi_y \neq 0$ shear

$$\lambda_{ijkl} u_{ij} u_{kl} \geq 0$$

$$= \lambda_{xyxy} + \lambda_{xyyx} + \lambda_{yxxy} + \lambda_{yxyx} = 4\mu \rightarrow \mu \geq 0$$

② $u_{ij} = \delta_{ij}$



expansion/ compression

$$\lambda_{ijkl}\delta_{ij}\delta_{kl} \geq 0$$

$$\hookrightarrow \tilde{K} \underbrace{\delta_{ij}\delta_{ij}}_{tr(I)=3} \delta_{kl}\delta_{kl} + \mu \delta_{ij}\delta_{kl}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) = 9\tilde{K} + 6\mu \geq 0$$

or $\tilde{K} \geq -\frac{2}{3}\mu$.

Write $K = \tilde{K} + \frac{2}{3}\mu$ so $K \geq 0$.

$$\lambda_{ijkl} = K \delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}\right)$$

bulk modulus

$\downarrow$

$\delta_{ij}\delta_{kl}$ = projection onto spin 0/ trace

shear modulus.

$\hookrightarrow$ projection onto spin 2 shearing solid/ volume-preserving deform.

Rotation invariance: 3x3 matrix = (spin 1) $\otimes$ (spin 1)

vector

= (spin 0) $\oplus$ $\boxed{\text{(spin 1)}}$ $\oplus$ (spin 2)

trace

?? antisymmetric → global rotation of solid.

traceless symmetric

$$\lambda_{ijkl} u_{kl} = K \delta_{ij}\, tr(u)$$