PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 2
Invariant building blocks

August 30

Lagrangian effective theory:

for one DOF $x(t)$:

usually ignore

$$S[x(t)] = \underbrace{\int dt}_{\text{locality in time!}} \, L(x, \dot{x}, \cancel{\ldots})$$

The EOMs: principle of least action

$$\frac{\delta S}{\delta x(t)} = 0 = \frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) \quad [\text{if } L(x, \dot{x}) \text{ only}]$$

Goal today: constrain L based on symmetry

→ write L in terms of invariant building blocks

↑ under symmetry

Result: most general possible $\underbrace{L(\text{inv BBs})}_{\text{effective theory.}}$

Example: suppose translation invariance
physics "same" if $x(t) \to x(t) + c$.

$$S[x(t)] = \underbrace{S[x(t)+c]}_{}.$$

$L(x, \dot{x}) = L(x+c, \dot{x}) \to L$ independent of x.

$$\frac{dL}{dc} = 0 = 1 \cdot \frac{\partial L}{\partial x} + 0 \cdot \frac{\partial L}{\partial \dot{x}} = 0, \qquad \frac{\partial L}{\partial x} = 0.$$

Solved by $L = f(\dot{x})$

Taylor expand?

$$f(\dot{x}) = a + b\dot{x} + c\dot{x}^2 + \cdots$$

lowest order eff. th.

Set $a = 0$:

$L = L_0 + a$

$S = S_0 + \underbrace{a(T-0)}_{\text{const.}}$

$$\frac{\delta S}{\delta x} = \frac{\delta S_0}{\delta x}$$

Set $b = 0$:

Suppose $L = L_0 + \frac{d}{dt}\Phi$ ($\Phi \to bx$)

$$S = \int_0^T dt \left(L_0 + \frac{d\Phi}{dt}\right)$$

$$= S_0 + \underbrace{\Phi(T) - \Phi(0)}_{\text{bdy term, can't affect } \frac{\delta S}{\delta x(t)},\ 0<t<T}$$

$L = c\dot{x}^2 + \cdots$

Call $c = \frac{m}{2}$ (mass), get $L = \frac{1}{2} m \dot{x}^2 + \cdots$

So far: $L = \frac{1}{2} m\dot{x}^2 - V(x)$ [relax trans. inv].

$$\frac{\delta S}{\delta x} = \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = -\frac{\partial V}{\partial x} - \frac{d}{dt}(m\dot{x})$$

$$m\ddot{x} = -\frac{\partial V}{\partial x} = F \quad \text{(conservative } F\text{)}$$

In general $L \neq$ kin − pot energy

In most eff th, multiple deg of freedom:

Principle of least action: $S[\underbrace{x_1(t), \ldots, x_n(t)}_{x_i(t) \; (\vec{x}(t))}]$, extremal on phys. traj.

implicit: $i = 1, \ldots, n$
dangling

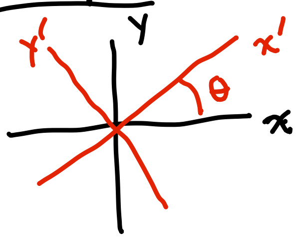
So write $S[x_i(t)]$.

Generalizing mult var calculus:

PoLA $\rightarrow$ $\frac{\delta S}{\delta x_i(t)} = 0$ [implicitly true for $1, \ldots, n = i$]

$$= \frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) \qquad \left[S = \int dt\, L\right]$$

Example: rotation invariance in 2d.



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Build theory (L) by find. invariant BBs.

Helpful: θ small:

$$\left.\begin{aligned} x &\to x + \theta y \\ y &\to y - \theta x \end{aligned}\right\} \qquad \begin{aligned} \cos\theta &= 1 + \cdots \\ \sin\theta &= \theta + \cdots \end{aligned}$$

What $f(x,y)$ invariant?

$$\frac{df}{d\theta} = \underbrace{\frac{dx}{d\theta}}_{=y} \frac{\partial f}{\partial x} + \underbrace{\frac{dy}{d\theta}}_{=-x} \frac{\partial f}{\partial y} = 0 \qquad \underline{y \frac{\partial V}{\partial x} - x \frac{\partial V}{\partial y} = 0}$$

Solve by method of characteristics:

$$\left.\begin{aligned} \frac{dx}{d\theta} &= y \\ \frac{dy}{d\theta} &= -x \end{aligned}\right\} \qquad \frac{dx}{dy} = -\frac{y}{x}$$

$x\,dx + y\,dy = 0$ integration const.

$\frac{1}{2}(x^2+y^2) = C$

indep. of θ.

Since x^2+y^2 ind. of θ, thus invariant BBs.

integration const.

Repeat arg:

$$\left.\begin{aligned} \dot{x} &\to \dot{x} + \theta \dot{y} \\ \dot{y} &\to \dot{y} - \theta \dot{x} \end{aligned}\right\} \qquad \dot{x}^2 + \dot{y}^2 \text{ is also invariant.}$$

Missing one!

$$\underbrace{L(x, y, \dot{x}, \dot{y})}_{4 \text{ param}} = L(x+\theta y, y-\theta x, \cdots)$$

4 param $- 1$ (cont. sym.) $= 3$.

Missing invariant: $x\dot{y} - y\dot{x}$

Our eff. th:

$$L\left(x^2+y^2,\ \dot{x}^2+\dot{y}^2,\ \underbrace{\cancel{x\dot{y}-y\dot{x}}}_{\text{“}=L_z\text{”}}\right)$$

(angular mom.)

(Lec 3: Noether's Thm)

Often useful: switch to polar coords

$$x = r\cos\theta \qquad \dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$$

$$y = r\sin\theta \qquad \dot{y} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}$$

$$\left.\begin{aligned} x^2+y^2 &= r^2 \\ \dot{x}^2+\dot{y}^2 &= \dot{r}^2 + r^2\dot{\theta}^2 \end{aligned}\right\} L(r^2,\ \dot{r}^2 + r^2\dot{\theta}^2)$$

Typically:

$$\approx \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

central force problem.