PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 20

Sound waves in solids

October 13

EFT for a solid: $\sigma^I(x_i, t) = \delta^I_i (x_i - \phi_i)$

infinitesimally small displacement.

elastic moduli

$$\mathcal{L} = \frac{\rho}{2} \partial_t \phi_i \partial_t \phi_i - \frac{1}{2} \underbrace{\lambda_{ijk\ell}} \partial_i \phi_j \partial_k \phi_\ell$$

for isotropic solid: $\lambda_{ijk\ell} = \underbrace{K \delta_{ij} \delta_{k\ell}}_{\substack{\text{compression} \\ \text{(volume changing)}}} + \underbrace{\mu \left(\delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk} - \frac{2}{3} \delta_{ij} \delta_{k\ell} \right)}_{\substack{\text{shear} \\ \text{(volume preserving)}}}$

$K, \mu \geq 0 \rightarrow$ ensures solids are stable.

Today: normal modes of solid.

EOMs: $$0 = \cancel{\frac{\partial \mathcal{L}}{\partial \phi_i}} - \partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t \phi_i)} - \partial_j \frac{\partial \mathcal{L}}{\partial(\partial_j \phi_i)}$$

$$0 = -\rho \partial_t^2 \phi_i + \partial_j \left[\lambda_{jik\ell} \partial_k \phi_\ell \right]$$

used $\lambda_{ijk\ell} = \lambda_{jik\ell} = \lambda_{k\ell ij}$

Plug in: plane wave ansatz: $\phi_i = a_i e^{i(\vec{k}\cdot\vec{x} - \omega t)}$

$$0 = \rho\omega^2 a_i - \underline{\lambda_{jikl}} k_j k_k a_l$$

For isotropic solid: $\vec{k} = \begin{pmatrix} 0 \\ 0 \\ k \end{pmatrix}$.

$$\rho\omega^2 a_i = \left[K\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \tfrac{2}{3}\delta_{ij}\delta_{kl}\right)\right] k_j k_k a_l$$

$$= K k_i k_l a_l + \mu k_i k_l a_l + \mu a_i k^2 - \tfrac{2}{3}\mu k_i k_l a_l = M_{il}(k) a_l$$

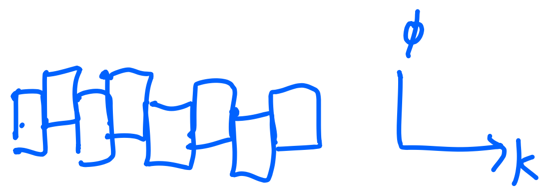
$$\downarrow$$

$$\rho\omega^2 \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{pmatrix} \boxed{\mu k^2} & 0 & 0 \\ 0 & \boxed{\mu k^2} & 0 \\ 0 & 0 & \boxed{(K + \frac{4}{3}\mu) k^2} \end{pmatrix} \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

transverse $\vec{a}, \vec{\phi} \perp \vec{k}$: S-waves

$$\omega = \pm v_s k$$

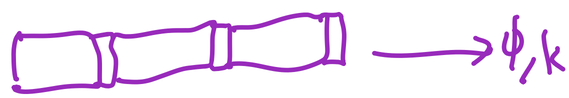
$$v_s = \sqrt{\frac{\mu}{\rho}}$$



longitudinal $\vec{a}, \vec{\phi} \parallel \vec{k}$: P-waves

$$\omega = \pm v_p k$$

$$v_p = \sqrt{\frac{K + \frac{4}{3}\mu}{\rho}}$$



Bulk solids have 2 types of sound (S & P):

A quantitative prediction:

$$\frac{v_p}{v_s} \geq \sqrt{\frac{4}{3}}. \quad \rightarrow \text{b/c } K, \mu \geq 0.$$

For typical crystalline solid: estimate v_s, v_p:

- $\rho \simeq \frac{m_{atom}}{d_{atom}^3} \sim \frac{10^{-25}\ kg}{(3\times10^{-10}\ m)^3} \sim 3\times10^3\ kg/m^3$

[compare to H_2O: $10^3\ kg/m^3$]

- estimate elastic modulus:

$$K, \mu \sim \frac{E_{bond}}{d_{atom}^3} \sim \frac{10^{-18}\ J}{(3\times10^{-10}\ m)^3}$$

$$\sim 3\times10^{10}\ J/m^3 \sim 3\times10^{10}\ Pa$$

$\sim 3\times10^5$ atm.

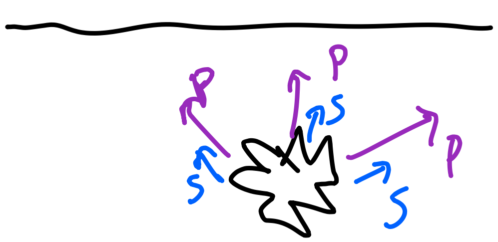
very hard to shear/compress solids.

$$\left[[v^2] = \frac{[\mu]}{[\rho]} \quad [\mu] = [\rho][v]^2 = \frac{[energy]}{[vol]} \right]$$

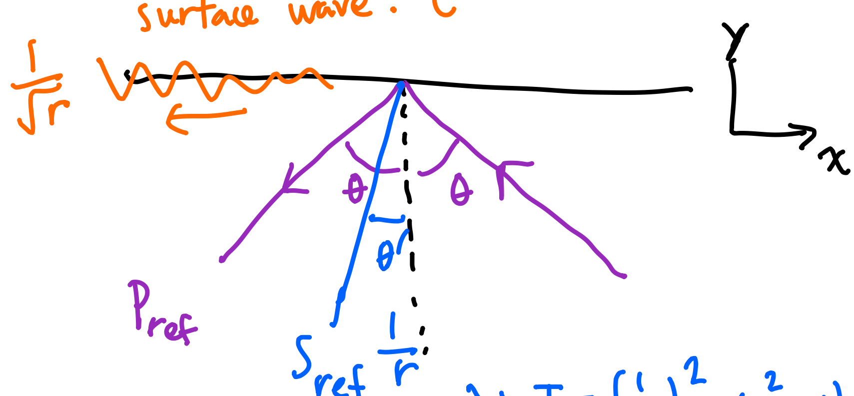
So speed of sound:

$$v \sim \sqrt{\frac{K}{\rho}} \sim 3\times10^3\ m/s = \underbrace{3\ \frac{km}{s}}_{10\times \text{ faster in air}}$$

Application: earthquakes



surface wave: $e^{ik_x x - i\omega t - \kappa |y|}$



$\hookrightarrow I = \left(\frac{1}{r}\right)^2 \cdot \underbrace{r^2}_{Area} \sim 1$

t-translation

$\downarrow$

$\omega = v_p k_p = v_s k_s \rightarrow k_s > k_p$

x-translation:

$\hookrightarrow k_{p,x} = k_{s,x} \longrightarrow k_{s,y} > k_{p,y}$