PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 21

Hamiltonian mechanics

October 16

Why Lagrangian mechanics?

1. deal w/ new configuration spaces (e.g. rigid body) — 1. even more abstract... phase space
2. easy to implement symmetries — 2. symmetries have algebraic structure
3. Noether's Thm — 3. symmetry $\Longleftrightarrow$ cons. law (if and only if)

Now: Hamiltonian mechanics.

Vision: Hamiltonian mechanics

intrinsically first order EOMs in t:

$$\dot{\xi}^\alpha = f^\alpha(\xi)$$

ξ's = coordinates on phase space (index $\alpha\beta\gamma\ldots$)

Today: Hamiltonian structure from Lagrangian mechanics:

Suppose $S[x_i] = \int dt\, L(x_i, \dot{x}_i)$

$$\frac{\delta S}{\delta x_i} = 0 = \frac{\partial L}{\partial x_i} - \frac{d}{dt}\boxed{\frac{\partial L}{\partial \dot{x}_i}}$$

call this p_i (generalized momentum)

$$\frac{\partial L}{\partial \dot{x}_i} = p_i$$

$$\frac{\partial L}{\partial x_i} = \dot{p}_i$$

Now we need $\dot{x}_i = ?$ invert $\frac{\partial L}{\partial \dot{x}_i} = p_i$ to get $\dot{x}_i = f_i(x,p)$

Good way to do this: Legendre transform!

$L(x,\dot{x})$

$$dL = \frac{\partial L}{\partial x_i} dx_i + \frac{\partial L}{\partial \dot{x}_i} d\dot{x}_i = \frac{\partial L}{\partial x_i} dx_i + \underbrace{p_i\, d\dot{x}_i}_{d(p_i \dot{x}_i) - \dot{x}_i dp_i}$$

$$d\underbrace{(L - p_i \dot{x}_i)}_{\text{define: } -H(x_i, p_i)} = \underbrace{\frac{\partial L}{\partial x_i}}\, dx_i - \underbrace{\dot{x}_i}\, dp_i$$

Hamiltonian

$$\dot{x}_i = \frac{\partial H}{\partial p_i}$$

$$-\frac{\partial H}{\partial x_i} = \frac{\partial L}{\partial x_i} = \dot{p}_i$$

Recap: start w/ $L(x_i, \dot{x}_i)$ $(i = 1, \ldots n)$

1. Define $H = \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} - L$
2. $p_i = \frac{\partial L}{\partial \dot{x}_i}$, and write $H(x_i, \dot{x}_i) \to H(x_i, p_i)$
3. Euler-Lagrange eqns $\longrightarrow$ Hamilton's equations:

$$\frac{\partial H}{\partial p_i} = \dot{x}_i, \quad -\frac{\partial H}{\partial x_i} = \dot{p}_i$$

intrinsically first order.

phase space: (x_i, p_i)

$2n$-dimensional

Example: harmonic oscillator:

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

1, 2

$$p = \frac{\partial L}{\partial \dot{x}} = m \dot{x}$$

$$H = p\dot{x} - L = p \cdot \frac{p}{m} - \left[\frac{m}{2} \left(\frac{p}{m} \right)^2 - \frac{k}{2} x^2 \right] = \frac{p^2}{2m} + \frac{kx^2}{2}$$

3 Hamilton's equations:

$$\frac{\partial H}{\partial p} = \dot{x} = \frac{p}{m} \quad \text{and} \quad -\frac{\partial H}{\partial x} = -kx = \dot{p}$$

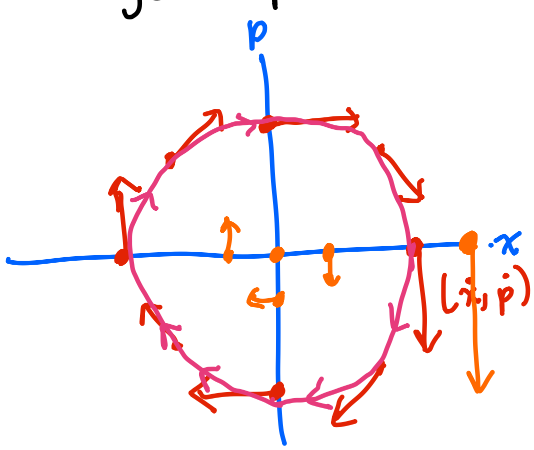
Convert back into single equation for x:

$$\dot{p} = -kx = \frac{d}{dt}(m\dot{x}) \longrightarrow \underbrace{m\ddot{x} = -kx.}_{\text{Euler-Lagrange}}$$

Euler-Lagrange → nothing new

Advantage: picture of dynamics on phase space:

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}$$



- draw Hamiltonian vector field
- physical trajectories follow vector field.

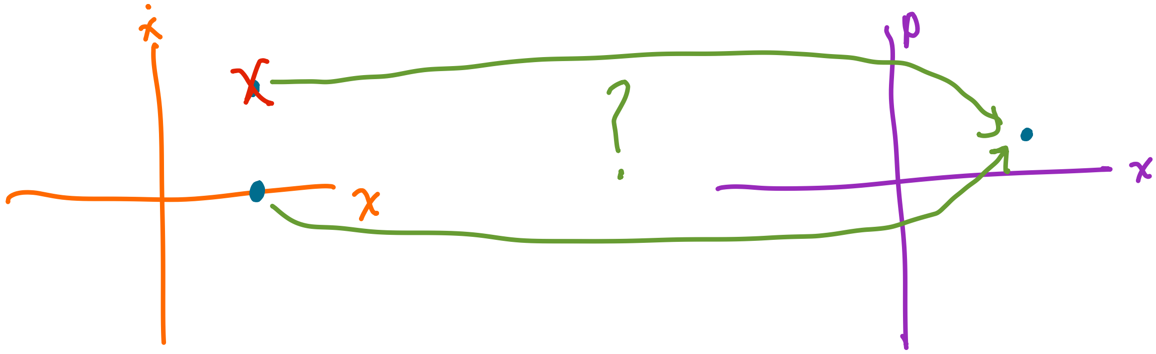
lec 22, 23: mathematical rigidity to these flows.

Is it possible always to find H?

Need legendre transformation to exist.

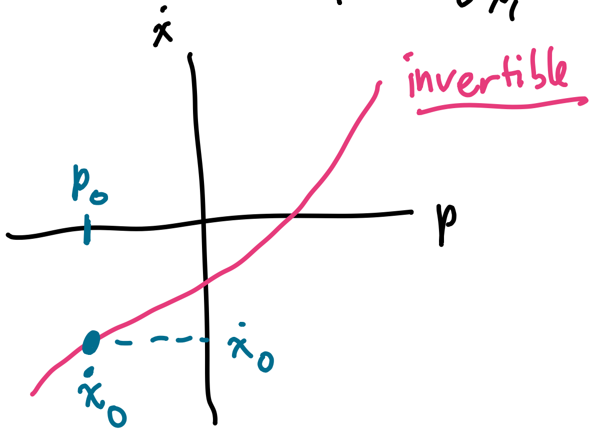
$$(x_i, \dot{x}_i) \xrightarrow{\text{Legendre}} (x_i, p_i) \qquad \left(p_i = \frac{\partial L}{\partial \dot{x}_i} \right)$$

config space → phase space



$p_i d\dot{x}_i \longrightarrow -\dot{x}_i dp_i \longrightarrow p_i d\dot{x}_i$

So we need $p_i = \dfrac{\partial L}{\partial \dot{x}_i}$ to have unique solutions.



$$p_i = \frac{\partial L}{\partial \dot{x}_i}$$

$$\downarrow$$

$$\frac{\partial^2 L}{\partial \dot{x}_i \partial \dot{x}_j} \geq 0$$

positive semidefinite

On flip side: if we start w/ $H(x_i, p_i)$, only comes from Lagrangian if $\dfrac{\partial^2 H}{\partial p_i \partial p_j}$ are positive semidefinite.

$$f_i \frac{\partial^2 H}{\partial p_i \partial p_j} f_j \geq 0.$$

for any f_i, at all points in phase space