PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 22
Poisson brackets

October 18

Review: Hamiltonian mechanics

$$-\frac{\partial H}{\partial x_i} = \dot{p}_i \qquad \frac{\partial H}{\partial p_i} = \dot{x}_i$$

explicit first order dynamics on phase space

By Legendre transform:

$$p_i = \frac{\partial L}{\partial \dot{x}_i} \quad \text{and} \quad H = p_i \dot{x}_i - L,$$ expressed in terms of x_i & p_i

Invert logic! Legendre from $H \to L$:

$$L = p_i \dot{x}_i - H(x,p)$$

principle of least action: $S = \int dt\, L = \int dt\, [p_i \dot{x}_i - H]$.

$$\left.\begin{aligned} \frac{\delta S}{\delta p_i} &= \frac{\partial L}{\partial p_i} = 0 = \dot{x}_i - \frac{\partial H}{\partial p_i} \\ \frac{\delta S}{\delta x_i} &= \frac{\partial L}{\partial x_i} - \frac{d}{dt} p_i = 0 = -\frac{\partial H}{\partial x_i} - \dot{p}_i \end{aligned}\right\} \text{Hamilton's eqns.}$$

Abstraction: work with phase space coordinates

$$\xi^\alpha = (x_i, p_i)$$

$\alpha = 1, \dots, 2n \qquad i = 1, \dots, n$

Define symplectic form:

$$\omega_{\alpha\beta} = \begin{array}{cc} x & p \\ \multicolumn{2}{c}{\left(\begin{array}{c|c} 0 & -\mathbb{1} \\ \hline \mathbb{1} & 0 \end{array}\right)} \end{array}$$

$n \times n$ identity matrix

$$\omega_{x_i p_j} = -\delta_{ij} \qquad \omega_{p_i x_j} = \delta_{ij}.$$

$\alpha = (x_1, \dots, x_n, p_1, \dots, p_n)$ $\qquad \omega_{p_i x_j} \to \omega_{(n+i)j}$

$$S = \int dt \left[p_i \dot{x}_i - H\right] = \int dt \left[\frac{p_i \dot{x}_i}{2} + \cancel{\frac{d}{dt}\left(\frac{p_i x_i}{2}\right)} - \frac{\dot{p}_i x_i}{2} - H\right]$$

$$= \int dt \left[\frac{1}{2}\omega_{\alpha\beta}\xi^\alpha \dot{\xi}^\beta - H(\xi)\right]$$

Note: $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$. Arrive at

$$\frac{\delta S}{\delta \xi^\alpha} = 0 = \frac{\partial L}{\partial \xi^\alpha} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\xi}^\alpha} = \left[\frac{1}{2}\omega_{\alpha\beta}\dot{\xi}^\beta - \frac{\partial H}{\partial \xi^\alpha}\right] - \frac{1}{2}\omega_{\beta\alpha}\frac{d}{dt}\xi^\beta$$

or: $$\omega_{\alpha\beta}\dot{\xi}^\beta = \frac{\partial H}{\partial \xi^\alpha}$$

Notice $\omega_{\alpha\beta}$ is invertible:

$$(\omega^{-1})_{\alpha\beta} = \omega^{\alpha\beta} = \left(\begin{array}{c|c} 0 & \mathbb{1} \\ \hline -\mathbb{1} & 0 \end{array}\right)$$

Re-write EOM: $$\boxed{\dot{\xi}^\alpha = \omega^{\alpha\beta}\frac{\partial H}{\partial \xi^\beta}}$$

shorthand:

$$\dot{x}_i = \frac{\partial H}{\partial p_i}$$

$$\dot{p}_i = -\frac{\partial H}{\partial x_i}$$

Define Poisson bracket:

$$\{f, g\} = \omega^{\alpha\beta}\frac{\partial f}{\partial \xi^\alpha}\frac{\partial g}{\partial \xi^\beta}$$

old coords

$$= \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x_i}$$

If $f(\xi)$ has no explicit t-dep:

$$\frac{df}{dt} = \frac{\partial f}{\partial \xi^\alpha}\dot{\xi}^\alpha = \frac{\partial f}{\partial \xi^\alpha}\omega^{\alpha\beta}\frac{\partial H}{\partial \xi^\beta} = \boxed{\{f, H\} = \frac{df}{dt}}$$

coord. ind. EOM.

Now justify this w/o Lagrangians:

① Hamiltonian mechanics on phase space M (2n-dim. manifold) (even)

② Add symplectic form $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$

- ω^{-1} exists everywhere. (PB defined)
- (lec 23) closed: [technical]

$$\partial_\alpha \omega_{\beta\gamma} + \partial_\beta \omega_{\gamma\alpha} + \partial_\gamma \omega_{\alpha\beta} = 0.$$

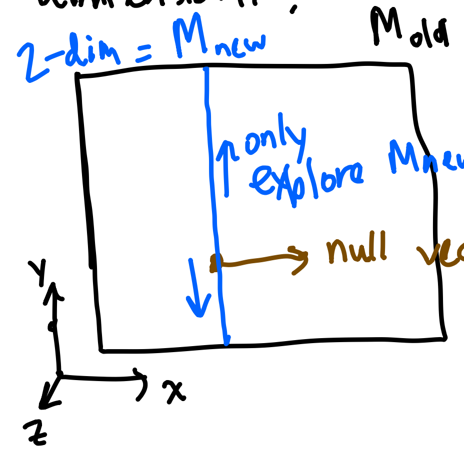
Comment: (M, ω) is called symplectic manifold.

③ Given any function H (Hamiltonian), generate "flows on phase space" (EOMs) by:

$$\dot{f} = \{f, H\} = \omega^{\alpha\beta} \partial_\alpha f \, \partial_\beta H.$$

- why phase space even-dimensional?

ω^{-1} doesn't exist otherwise.



$\dot{x} = \{x, H\} = 0$ for any H.

(assume ω^{-1} was defined directly and ω doesn't exist)

- "coordinate-independent": $\dot{f} = \{f, H\}$.

↑ more lec. 24

$\tilde{\xi}^a$ = new coords:

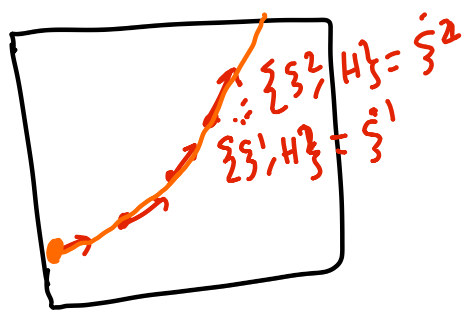
$\dot{\tilde{\xi}}^a = \{\tilde{\xi}^a, H\}$ same flow as

$\dot{\xi}^\alpha = \{\xi^\alpha, H\}$

- Noether Thm? Suppose found Q w/ $\{Q, H\} = 0$.
  Then $\dot{Q} = 0$ i.e. Q is conserved.
  (But here started w/ Q, not symmetry? Lec 25)

- lec 23: $\{f, g\}$ is another function...
  can feed back into PBs.
  This generates Lie algebra.

- flows in phase space.



- deep connection to QM:
  define canonical coordinates x_i, p_i: $\{x_i, p_j\} = \delta_{ij}$ $\longrightarrow$ operators $\hat{x}_i, \hat{p}_i$: $\hat{p}_i = -i\hbar \dfrac{\partial}{\partial x_i}$

  $[\hat{x}_i, \hat{p}_j] = \hat{x}_i \hat{p}_j - \hat{p}_j \hat{x}_i = i\hbar \delta_{ij}$.

More generally:

$$\underbrace{\{f, g\}}_{\text{classical}} \longrightarrow \underbrace{\frac{1}{i\hbar}[f, g]}_{\text{quantum}}$$

$$\frac{df}{dt} = \{f, H\} \longrightarrow i\hbar \frac{df}{dt} = [f, H]$$

Heisenberg EOM! solved by $f(t) = e^{iHt/\hbar} \cdot f e^{-iHt/\hbar}$