PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 23
Poisson brackets as a Lie algebra

October 20

Last time: Hamiltonian mechanics

① Phase space (coords ξ^α, "optional"), w/ $\alpha = 1, \dots, 2n$ (even)

② Symplectic form $\omega_{\alpha\beta}$ / Poisson bracket $\omega^{\alpha\beta}$: $= -\omega^{\beta\alpha}$

$$\{f, g\} = \omega^{\alpha\beta} \frac{\partial f}{\partial \xi^\alpha} \frac{\partial g}{\partial \xi^\beta}$$

and $(\omega^{\alpha\beta})^{-1} = \omega_{\alpha\beta}$, inverse exists everywhere on phase space

and Jacobi identity (soon) → [Symp. form closed]

$$\{f, g\} = -\{g, f\}$$

then: ③ for any function H (Hamiltonian), generates dynamics:

$$\dot{f} = \{f, H\} \qquad [\text{for any } f].$$

Claim: Poisson bracket EOM only OK if Jacobi identity:

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0.$$

for all f, g, h.

Proof: If f, g functions, $\{f, g\}$ also function...

$$\frac{d}{dt}\{f,g\} = \{\{f,g\},H\}$$ by ③.

But: also use chain rule:

$$\frac{d}{dt}\{f,g\} = \{\dot{f},g\} + \{f,\dot{g}\}$$

$$= \{\{f,H\},g\} + \{f,\{g,H\}\}$$

$$= -\{f,\{H,g\}\}$$

$$= \{\{H,g\},f\}.$$

So: $\{\{f,H\},g\} + \{\{H,g\},f\} - \{\{f,g\},H\} = 0.$

$$\boxed{\{\{f,H\},g\} + \{\{H,g\},f\} + \{\{g,f\},H\} = 0.}$$ Jacobi

f, g, H are arbitrary.

No preferred ξ^α for which ~~$\{\xi^\alpha, H\} = \dot{\xi}^\alpha$ but $\{f,H\} \neq \dot{f}$~~

Define a Lie algebra by a bilinear on vector space:

① $f, g \in$ space (func. on phase space) $\longrightarrow$ $\{f,g\} \in$ space

② bilinearity: $\{c_1 f_1 + c_2 f_2, g\} = c_1\{f_1,g\} + c_2\{f_2,g\}$
if c_1, c_2 const.

③ $\{f,g\} = -\{g,f\}$

④ Jacobi identity.

If vector space = space of function on phase space...
Poisson brackets form a Lie algebra.

Define a m-dimensional (sub)algebra, F^a $(a=1,\ldots,m)$

such that $\{F^a, F^b\} = f^{abc}F^c$

↑ structure constants

$$f^{abc} = -f^{bac}.$$

Example 1: $\mathbb{R}^2$ phase space (x, p): $F^1 = x$, $\underbrace{F^2 = p, \ F^3 = 1}_{\downarrow}$

Subalgebra of (sub)algebra...

$$\{x, p\} = 1 \longrightarrow \begin{array}{l} \{F^1, F^2\} = F^3 \\ \{F^{1,2}, F^3\} = 0. \end{array}$$

so $f^{123} = -f^{213} = 1$, $f^{132} = 0$ etc.

Recall: given a Hamiltonian H, Q conserved if $\{Q, H\} = \dot{Q} = 0$.

Claim: if Q_1, Q_2 are conserved, then $\{Q_1, Q_2\}$ conserved.

$$\frac{d}{dt}\{Q_1, Q_2\} = \{\{Q_1, Q_2\}, H\}$$

$$= -\{\overset{0}{\cancel{\{Q_2, H\}}}, Q_1\} - \{\overset{= -\{Q_1, H\} = 0.}{\cancel{\{H, Q_1\}}}, Q_2\} = 0.$$

Conclusion: conserved quantities of any H form a (sub)algebra.

Example 2: 3d, angular momentum algebra:

Define $L_i = \varepsilon_{ijk} x_j p_k$ ⤷ $L_x = y p_z - z p_y$

$L_y = z p_x - x p_z$

$L_z = x p_y - y p_x$.

Calculate $\{L_i, L_j\} = ?$

- since $\{f, g_1 g_2\} = \{f, g_1\} g_2 + g_1 \{f, g_2\}$.
- $\{L_i, x_j\} = \cancel{\frac{\partial L_i}{\partial x_k} \frac{\partial x_j}{\partial p_k}} - \frac{\partial L_i}{\partial p_k} \frac{\partial x_j}{\partial x_k}$

  $= -\varepsilon_{imk} x_m \delta_{jk} = \varepsilon_{ijm} x_m$
- Similarly: $\{L_i, p_j\} = \varepsilon_{ijm} p_m$

You can show that $\{L_i, L_j\} = \varepsilon_{ijk} L_k$

Thus, 3 components of L_i form a subalgebra.

Physical: possible to conserve L_x, L_y, L_z but not other functions.

Classical: $x(t)$, $p(t)$: $\dot{x} = \{x, H\}$

QM: $\hat{x}(t)$, $\hat{p}(t)$ operators on $|\psi(t)\rangle$.

$$\frac{d\hat{x}(t)}{dt} = \frac{1}{i\hbar}[\hat{x}, H].$$

↳ $\hat{x}\hat{p} \neq \hat{p}\hat{x}$ makes QM exciting.

but $|\psi(t)\rangle$ different.

↳ $\psi(x,t)$

What's the most general H that conserves L_i?
(for "one particle", phase space $\mathbb{R}^6$ (x_i, p_i)).

Solve: $\{L_i, H\} = 0$.

$$\hookrightarrow \; = \varepsilon_{ijk}\left[x_k \frac{\partial H}{\partial x_j} + p_k \frac{\partial H}{\partial p_j}\right] = 0.$$

Solutions? ① $H = x_i x_i$, since

$$\varepsilon_{ijk} x_k (2x_j) = 2\varepsilon_{ijk} x_j x_k = 0.$$

similarly, ② $H = p_i p_i$ ③ $H = p_i x_i$

So most general $H(\underbrace{x_i x_i, p_i p_i, p_i x_i}_{\text{inv. BBs}})$ that obeys $\{L_i, H\} = 0$.

↓

rotation invariance?! (Lec 25)