PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 24
Canonical transformations

October 23

Recall: in Lagrangian mechanics,

symmetry:
e.g. $x(t) \to x(t) + \epsilon$
and $S = \int dt\, L$ invariant

$\xrightarrow{\text{Noether's Thm}}$

conserved quantity:
$\dot{Q} = 0$
(evaluated on EOMs)

Hamiltonian mechanics?

? (today)

$\xleftrightarrow{\text{Lec 25}}$

$\dot{Q} = 0$ if $\{Q, H\} = 0$
Q's have algebraic structure
$\{Q_1, Q_2\}$ conserved.

Structure of Hamiltonian mechanics:

① phase space → ② Poisson bracket → ③ Hamiltonian.

Symmetry = coordinate change $\xi^\alpha \to \eta^\alpha(\xi)$ that?
(x,p)

– leave H unchanged. ⤳ $H(x,p) = H(X,P)$
$p^2/2m = P^2/2m$

– leave PBs unchanged.

do this first b/c "PB > H"

A <u>canonical transformation</u> is $\xi^\alpha \to \eta^\alpha(\xi)$ that leaves PB invariant: $\{\xi^\alpha, \xi^\beta\} = \{\eta^\alpha, \eta^\beta\}$

Example 1: Phase space $\xi = (x, p)$, $\{x, p\} = 1$.

η's

A) $X = x + p$, $P = p$: $\{X, P\} = \{x+p, p\} = \{x, p\} + \cancel{\{p, p\}}$

$= CT$

B) $X = x^2$, $P = p$: $\{X, P\} = \{x^2, p\} = 2x\{x, p\} = 2x \neq 1.$

$\neq CT$

$\{f, g\} = \omega^{\alpha\beta}\partial_\alpha f \,\partial_\beta g$

What CT's possible? Start by asking:

<u>infinitesimal CT</u>: $\xi^\alpha \to \xi^\alpha + \varepsilon\theta^\alpha$ (ε = infinitesimal)

What's most general form of θ^α?

$$\cancel{\{\xi^\alpha, \xi^\beta\}} = \{\xi^\alpha + \varepsilon\theta^\alpha, \xi^\beta + \varepsilon\theta^\beta\} = \cancel{\{\xi^\alpha, \xi^\beta\}} + \varepsilon\{\theta^\alpha, \xi^\beta\} + \varepsilon\{\xi^\alpha, \theta^\beta\} + \dots$$

So: $\{\xi^\alpha, \theta^\beta\} + \{\theta^\alpha, \xi^\beta\} = 0$

Recall: $\{f, g\} = \omega^{\alpha\beta}\dfrac{\partial f}{\partial \xi^\alpha}\dfrac{\partial g}{\partial \xi^\beta}$. For simplicity assume:

abused :(

canonical coordinates: $$\omega^{\alpha\beta} = \begin{array}{c} \begin{array}{cc} x_i & p_i \end{array} \\ \left(\begin{array}{c|c} 0 & \mathbb{1} \\ \hline -\mathbb{1} & 0 \end{array}\right) \end{array} \rightsquigarrow \{x_i, p_j\} = \delta_{ij}.$$

$$\{\xi^\alpha, g\} = \omega^{\alpha'\beta'}\frac{\partial \xi^\alpha}{\partial \xi^{\alpha'}}\frac{\partial g}{\partial \xi^{\beta'}} = \omega^{\alpha'\beta'}\delta^\alpha_{\alpha'}\partial_{\beta'} g = \omega^{\alpha\beta}\partial_\beta g$$

So: $\omega^{\alpha\gamma}\partial_\gamma\theta^\beta + \omega^{\delta\beta}\partial_\delta\theta^\alpha = 0.$

Recall: PBs have <u>inverse</u> (symplectic form): $\omega^{\alpha\beta}\omega_{\beta\gamma} = \delta^\alpha_\gamma$

$$(\omega^{\alpha\gamma}\partial_\gamma\theta^\beta + \omega^{\delta\beta}\partial_\delta\theta^\alpha)\,\omega_{\mu\alpha}\omega_{\beta\nu}$$

$$= \underbrace{(\cancel{\omega_{\mu\alpha}\omega^{\alpha\gamma}})}_{\delta^\gamma_\mu}\partial_\gamma\theta^\beta\omega_{\beta\nu} + \underbrace{(\cancel{\omega^{\delta\beta}\omega_{\beta\nu}})}_{\delta^\delta_\nu}\partial_\delta\theta^\alpha\omega_{\mu\alpha}$$

Define: $\theta_\mu = \omega_{\mu\alpha}\theta^\alpha$, since $\omega^{\alpha\beta}$, $\omega_{\alpha\beta}$ = const., then

$$= \partial_\mu \underbrace{\theta^\beta\omega_{\beta\nu}}_{= -\omega_{\nu\beta}\theta^\beta} + \partial_\nu\theta^\alpha\omega_{\mu\alpha} = -\partial_\mu\theta_\nu + \partial_\nu\theta_\mu = 0$$

Math Fact: up to ~~topology~~ ignore, all sol'ns to above: $\theta_\mu = \partial_\mu F$

$$\theta^\alpha = \omega^{\alpha\mu}\theta_\mu = \omega^{\alpha\mu}\partial_\mu F = \{\xi^\alpha, F\}.$$

General form of (infinitesimal) CT: $\xi^\alpha \to \xi^\alpha + \varepsilon\{\xi^\alpha, F\}$.

Example 2: (x,p) phase space.

Most general? $x \to x + \varepsilon\dfrac{\partial F}{\partial p}$ and $p \to p - \varepsilon\dfrac{\partial F}{\partial x}$.

Take: $F = p$ $\qquad \underbrace{x \to x + \varepsilon \quad \text{and} \quad p \to p.}_{\text{translation symmetry generated by } \underline{p}.}$

Spoiler: suppose this CT leaves H invariant: $\{p, H\} = 0$

$\hookrightarrow \dot{p} = 0$.

Example 3: time-translation. Take $H = H(\xi)$ (t-ind.)

There is a CT: $\xi^\alpha \to \xi^\alpha + \varepsilon\{\xi^\alpha, H\}$

$= \xi^\alpha + \varepsilon\dot{\xi}^\alpha \approx \xi^\alpha(\varepsilon)$

Since $\{\xi^\alpha, H\} = \dot{\xi}^\alpha$

Thus: $\xi^\alpha \underset{CT}{\rightarrow} \xi^\alpha(\varepsilon) \underset{CT}{\rightarrow} \xi^\alpha(2\varepsilon) \rightarrow \cdots \underset{CT}{\rightarrow} \xi^\alpha\left(\frac{t}{\varepsilon}\cdot\varepsilon\right) = \xi^\alpha(t)$

So $\xi^\alpha \xrightarrow[CT]{} \xi^\alpha(t)$ is a large CT.

Define: $\mathrm{ad}_F(g) = \{F, g\} = -\{g, F\}$.

Then: $\xi^\alpha(t) = \left(e^{-t\cdot \mathrm{ad}_H}\right)\xi^\alpha$

↳ $$\frac{d\xi^\alpha(t)}{dt} = (-\mathrm{ad}_H)\left[e^{-t\cdot\mathrm{ad}_H}\xi^\alpha\right] = -\mathrm{ad}_H\,\xi^\alpha(t) = \{\xi^\alpha(t), H\}.\ \checkmark$$

Analogue to "Heisenberg picture":

cf QM:

$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle$

$\langle\psi(t)|A|\psi(t)\rangle = \langle\psi(0)|A(t)|\psi(0)\rangle$

$A(t) = e^{iHt/\hbar} A e^{-iHt/\hbar}$

$\sim e^{-t\cdot\underbrace{\frac{1}{i\hbar}[H,\cdot]}_{=\mathrm{ad}_H}} A$

Continuous symmetry in Hamiltonian mech

= CTs (generated by function F)

$$\xi^\alpha \rightarrow \xi^\alpha + \varepsilon\{\xi^\alpha, F\}.$$

leave invariant:

① phase space ② Poisson bracket ③ Hamiltonian (?)

lec 25.

(Noether Thm)