PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 25

Noether's Theorem in Hamiltonian mechanics

October 25

Last time: canonical transformation (CT)

$$\xi^\alpha \to \eta^\alpha(\xi)$$

preserve Poisson bracket: $\{\xi^\alpha, \xi^\beta\} = \{\eta^\alpha, \eta^\beta\}$.

Infinitesimal CT $[\xi^\alpha \to \xi^\alpha + \varepsilon\,\theta^\alpha]$ generated by function F:

$$\xi^\alpha \to \xi^\alpha + \varepsilon\{\xi^\alpha, F\}, \quad \varepsilon \text{ infinitesimal.}$$

This CT is a symmetry if it leaves H invariant:

$$H(\xi^\alpha + \varepsilon\{\xi^\alpha, F\}) = H(\xi^\alpha).$$

$$\frac{dH_\varepsilon}{d\varepsilon} = \underline{0} = \frac{\partial H}{\partial \xi^\alpha}\{\xi^\alpha, F\} = \frac{\partial H}{\partial \xi^\alpha}\,\omega^{\alpha\beta}\frac{\partial F}{\partial \xi^\beta} = \underline{\{H, F\}}$$

If a symmetry generated by F, then

$$0 = \{H, F\} = -\frac{dF}{dt}. \quad \text{So } F \text{ is conserved quantity.}$$

Reverse logic!

If F is conserved, then $\{H, F\} = 0$.

CT generated by F leave H invariant $\to$ F generates sym!

Noether's Thm in Hamiltonian mech:

(continuous) symmetry: CT generated by F $\quad \xleftrightarrow{\text{if and only if}} \quad$ conserved quantity: $\dot{F} = \{F, H\} = 0$

QM: cont. symmetry:

$U = e^{-i\varepsilon F}$ (F Herm.)

$U^\dagger H U = H$

$H \approx H + i\varepsilon(FH - HF) + \cdots$

ε = infinitesimal

$\to [F, H] = 0$.

Hamiltonian mech: algebraic structure to symmetries:

F_1, F_2 conserved $\longrightarrow$ $\{F_1, F_2\}$ = conserved.

Example 1: translation symmetry.

Consider phase space $\mathbb{R}^{2n}$: (x_i, p_i) $\quad \{x_i, p_j\} = \delta_{ij}$ $\quad i = 1, \ldots, n$.

let $z = x_n$ and $s = p_n$.

Suppose that ① s is conserved: $\{s, H\} = 0$.

This means: $\{s, H\} = \frac{\partial s}{\partial x_i}\frac{\partial H}{\partial p_i} - \frac{\partial s}{\partial p_i}\frac{\partial H}{\partial x_i} = -\frac{\partial H}{\partial z} = 0$.

or ② H is independent of z. (conjugate variable to s)

If H independent of z, then symmetry: $z \to z + \varepsilon$.

This symmetry generated by s:

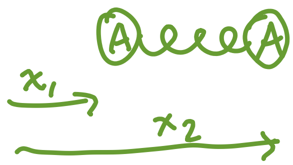
$$\xi^\alpha \to \xi^\alpha + \varepsilon \{\xi^\alpha, F\} \quad \text{and} \quad z \to z + \varepsilon \cancel{\{z, s\}}$$

Since H invariant:

① $\{H, s\} = 0 = -\dot{s}$ so s conserved.

Chain of logic can start either ① or ②

Example 2: diatomic molecule in 1d.



(also p_1, p_2)

I expect translation symmetry:

$x_1 \to x_1 + \varepsilon$, $p_1 \to p_1$

$x_2 \to x_2 + \varepsilon$, $p_2 \to p_2$

What F generate this CT? $\xi^\alpha \to \xi^\alpha + \varepsilon \{\xi^\alpha, F\}$.

$P = p_1 + p_2$ (total momentum)

$\{p_i, F\} = 0 = -\frac{\partial F}{\partial x_i}$, and $\{x_1 - x_2, F\} = 0$

so $\frac{\partial F}{\partial p_1} = \frac{\partial F}{\partial p_2}$, or $F(p_1 + p_2)$

$x_1 \to x_1 + \varepsilon$? $\{x_1, F\} = 1 = F'(p_1 + p_2)$, so $F = p_1 + p_2 = P$.

The most general H w/ this symmetry is:

made out of invariant BBs (functions f obey $\{f, P\} = 0$).

3 independent f's: $x_1 - x_2$, p_1, p_2: $H(p_1, p_2, x_1 - x_2)$

Hamiltonian mechanics makes manifest that

If we have k (independent, continuous) symmetries on $2n$-dim. phase space, "reduce" dim. of effective phase space $2(n-k)$

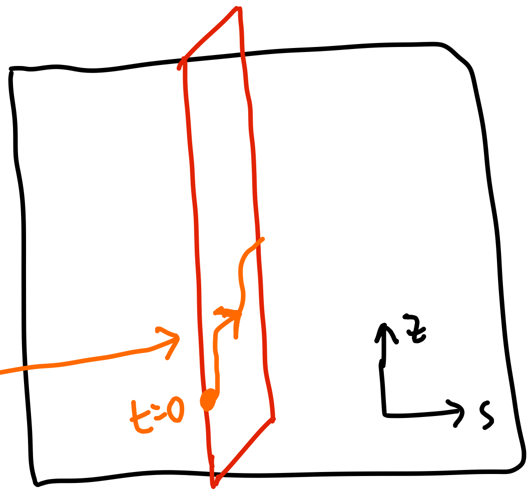
Example 1 (again): $\{s, H\}_{p_n} = 0$ meant that:

① $\dot{s} = 0$ (dynamics constrained to subspace w/ const. $s(0)$)

② $\dot{z} = +\frac{\partial H}{\partial s}(\underbrace{x_1, \ldots, x_{n-1}, p_1, \ldots, p_{n-1}}_{2(n-1)\text{ DOF}}, s)$ ← const. $\left[\frac{\partial H}{\partial z} = 0.\right.$

Just integrate for $z(t)$ at the end.

Phase space



2n-2-dim subspace: Hamiltonian dynamical system.

e.g. $\dot{x}_1 = \frac{\partial H}{\partial p_1}(\overset{2n-2}{\ldots}, s)$

$\downarrow$

$x_1(t)$

- $H_{red}(x_1, \ldots, p_{n-1}) = H(x_1, \ldots, p_{n-1}, s)$ ← const.
- $\{x_i, p_j\} = \delta_{ij}$ $(i, j = 1, \ldots, n-1)$ still valid/invertible PB.

Math: method of symplectic reduction