PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 26
Rigid body rotation in Hamiltonian mechanics

October 27

Rigid body rotation: configuration space $SO(3)$:

orthogonal R_{iI}, obeys $R_{iI}R_{iJ} = \delta_{IJ}$.

space frame (i) body frame. (I)

Lagrangian: $L = \frac{1}{2}\dot{R}_{iI}\dot{R}_{iJ}K_{IJ} + \Lambda_{IJ}(R_{iI}R_{iJ} - \delta_{IJ})$

(Lagrange multiplier)

In Hamiltonian mechanics?

① phase space → ② Poisson bracket → ③ Hamiltonian

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① phase space?

Naively: (coords) R_{iI} & (momenta)? P_{iI}

R_{iI} ↓ 3x3 matrix

$R_{iI} \in \underbrace{SO(3)}_{\text{3-dim}}$ So only 3 of P_{iI} & R_{iI} are actual DOF?

Write: $P_{iI} = R_{iJ} P_{JI}$:

Claim: $P_{JI} = -P_{IJ}$.

$$S = \text{"}\int dt \left[\tfrac{1}{2}\omega_{\alpha\beta}\xi^\alpha \dot{\xi}^\beta - H\right]\text{"}.$$

$$\to \int dt\left[\tfrac{1}{2} P_{iI}\dot{R}_{iI} - H\right] = \int dt\left[\tfrac{1}{2} \underline{R_{iJ}P_{JI}\dot{R}_{iI}} - H\right]$$

$= P_{JI}\Omega_{JI}$

If $R_{iI} \in SO(3)$, $R_{iJ}\dot{R}_{iI} = \Omega_{JI} = -\Omega_{IJ}$.

$\to$ Restrict to antisymmetric $P_{JI} = -P_{IJ}$.

So phase space: $R \in SO(3)$ and $P_{JI} = -P_{IJ} = \varepsilon_{JIK} L_K$

body frame ang. mom.

Euler!

② Poisson brackets:

Again, look at action: "$S = \int dt \tfrac{1}{2}\omega_{\alpha\beta}\xi^\alpha\dot{\xi}^\beta$"

if const!

$$\to \tfrac{1}{2} R_{iJ} P_{JI} \dot{R}_{iI} = \tfrac{1}{2} \underline{R_{iJ}}\, \varepsilon_{JIK} \underline{L_K}\, \underline{\dot{R}_{iI}}$$

$\omega_{\alpha\beta} \neq$ const.

Be more careful!

"$\lambda_\beta = \tfrac{1}{2}\omega_{\alpha\beta}\xi^\alpha$", but in general: $L = \lambda_\beta \dot{\xi}^\beta - H$.

Euler-Lagrange equations:

$$\frac{\partial L}{\partial \xi^\alpha} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\xi}^\alpha} = 0$$

$$= (\partial_\alpha \lambda_\beta)\dot{\xi}^\beta - \partial_\alpha H - \frac{d}{dt}\lambda_\alpha$$

$\frac{d}{dt}\lambda_\alpha = \partial_\beta \lambda_\alpha \dot{\xi}^\beta$

So: $\partial_\alpha H = (\partial_\alpha\lambda_\beta - \partial_\beta\lambda_\alpha)\dot{\xi}^\beta$

$= \omega_{\alpha\beta}$ $[\omega = d\lambda]$

So now use: $L = \tfrac{1}{2} R_{iJ}\varepsilon_{JIK} L_K \dot{R}_{iI} - H$:

Temporarily: $R_{iI} = \underbrace{\delta_{iI}}_{\text{identity}} + \underbrace{\varepsilon_{iIJ}}_{\text{rotate around } \phi_J \ldots} \phi_J$

Most general (infinitesimal) orthogonal: lec 9.

$$
\begin{aligned}
\text{“}\lambda_\beta \dot{\xi}^\beta\text{”} &= \frac{1}{2}\left(\delta_{iJ} + \varepsilon_{iJM}\phi_M\right) \varepsilon_{JIK} L_K \varepsilon_{iIN} \dot{\phi}_N \\
&= \frac{1}{2} \varepsilon_{JIK} \varepsilon_{JIN} L_K \dot{\phi}_N + \frac{1}{2} \varepsilon_{JIK} \phi_M L_K \left(\delta_{IJ}\delta_{MN} - \delta_{IM}\delta_{JN}\right) \dot{\phi}_N \\
&= \frac{1}{2} \cdot 2 \delta_{KN} L_K \dot{\phi}_N - \frac{1}{2} \varepsilon_{JIK} L_K \phi_I \dot{\phi}_J
\end{aligned}
$$

Read off: $\lambda_\beta \to \lambda_{\phi_I} = L_I - \frac{1}{2} \varepsilon_{IJK} L_K \phi_J$, $\lambda_{L_I} = 0.$

Then: $\omega_{\alpha\beta} = \partial_\alpha \lambda_\beta - \partial_\beta \lambda_\alpha$: neglect! $\phi \to 0.$

$$\omega_{L_I \phi_J} = \delta_{IJ} - \cancel{\frac{1}{2} \varepsilon_{IKJ} \phi_K}$$

$$\omega_{\phi_I \phi_J} = \underline{- \varepsilon_{IJK} L_K}$$

doesn't vanish if $\phi = 0$, keep!

$$
\omega_{\alpha\beta} = \begin{array}{c} \left(\begin{array}{c|c} -\varepsilon_{IJK} L_K & -\delta_{IJ} \\ \hline \delta_{IJ} & \end{array}\right) \\ \begin{array}{cc} \phi_I \quad & \quad L_I \end{array} \end{array}
$$

$\downarrow$

$$
\omega^{\alpha\beta} = \begin{array}{c} \left(\begin{array}{c|c} 0 & \delta_{IJ} \\ \hline -\delta_{IJ} & -\varepsilon_{IJK} L_K \end{array}\right) \\ \begin{array}{cc} \phi \quad & \quad L \end{array} \end{array}
$$

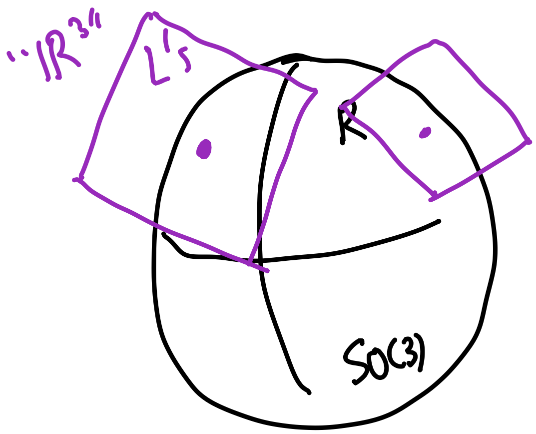
Physically interpret:

$$\{R_{i\,I}, L_J\} = R_{iK}\,\varepsilon_{KIJ}$$

$$\{L_I, L_J\} = \underline{-\varepsilon_{IJK} L_K}$$

relative minus sign! b/c this is body frame

Key point: phase space for rigid body best described in <u>non-canonical</u> coords: $\omega^{\alpha\beta} \neq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.



$\{L_I, L_J\} = -\varepsilon_{IJK} L_K$.

↳ "cotangent bundle of SO(3)"

③ Hamiltonian. Simple!

Rigid body rotation: left – SO(3) symmetry: Space frame indices contracted!

$$H(R_{iI}, L_I) = H(\underbrace{\cancel{R_{iI} R_{iJ}}}_{\delta_{IJ}}, L_I) = H(L_I)$$

Effective theory:

$$H(L_I) = \cancel{A} + \cancel{B_I} L_I + C_{IJ} L_I L_J + \cdots$$

ignore constant.

want time-reversal symmetry

Rotate our body frame axes so C is diagonal:

$$H = \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3}, \quad I\text{'s} = \text{moments of inertia,}$$

$$\dot{L}_I = \{L_I, H\} = \{L_I, L_J\}\frac{\partial H}{\partial L_J} = -\varepsilon_{IJK}\,\omega_K$$

Define $\omega_I = \frac{\partial H}{\partial L_I} = \frac{L_I}{I_I}$

$$\dot{L}_1 = L_2\omega_3 - L_3\omega_2 = (I_2 - I_3)\,\omega_2\omega_3$$
$$\dot{L}_2 = L_3\omega_1 - L_1\omega_3 = (I_3 - I_1)\,\omega_3\omega_1$$
$$\dot{L}_3 = L_1\omega_2 - L_2\omega_1 = (I_1 - I_2)\,\omega_1\omega_2$$

All of "hardness" of Euler → ② (Poisson brackets).

$\dot{L} = [L, \Omega]$: most general left-$SO(3)$ inv. dynamics on phase space, arbitrary H.

lec 10

More! HW9.