PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 27
Generating functions for canonical transformations

October 30

Hamiltonian mechanics: $F(x,p) \rightarrow \frac{dF}{dt} = \{F, H\}$.

① algebra of symmetry generators

② insight into solvable (integrable) vs. not solvable (chaotic) systems...

↙ rest of class...

First goal: what are the <u>best</u> coords on phase space?

each conservation law "removed" 2 dimensions from phase space...

This week: "finding conserved q's..."

Today: finding good "large!" canonical transformations.

$\xi^\alpha \rightarrow \eta^\alpha(\xi)$, where $\{\xi^\alpha, \xi^\beta\} = \{\eta^\alpha, \eta^\beta\}$

Lec 24: infinitesimal CT generated by $\xi^\alpha \rightarrow \xi^\alpha + \varepsilon\{\xi^\alpha, F\}$

↳ "$\eta^\alpha_{(s)} \rightarrow e^{s \cdot \mathrm{ad}_F} \xi^\alpha$"

Last time: $\omega_{\alpha\beta} = \partial_\alpha \lambda_\beta - \partial_\beta \lambda_\alpha$ [always possible, up to topological]

Idea: $\omega_{\alpha\beta}$ = invariant under $\xi \to \eta$ if $\{\cdot,\cdot\}$ is

$\xi \to \lambda_\alpha$, $\eta \to \theta_\alpha$

$$\omega_{\alpha\beta} = \partial_\alpha \lambda_\beta - \partial_\beta \lambda_\alpha = \partial_\alpha \theta_\beta - \partial_\beta \theta_\alpha$$

$$\hookrightarrow \omega_{\alpha\beta} - \omega_{\alpha\beta} = 0 = \partial_\alpha(\lambda-\theta)_\beta - \partial_\beta(\lambda-\theta)_\alpha$$

Solved by: $\lambda_\alpha - \theta_\alpha = \partial_\alpha F$

A la thermodynamics: $\lambda_\alpha d\xi^\alpha - \theta_\alpha d\eta^\alpha = dF = \frac{\partial F}{\partial \xi^\alpha} d\xi^\alpha$

$\hookrightarrow + \frac{\partial F}{\partial \eta^\alpha} d\eta^\alpha$

Assuming: $\xi^\alpha = (x_i, p_i)$ and $\eta^\alpha = (X_i, P_i)$,

$\{x_i, p_j\} = \delta_{ij}$ and $\{X_i, P_j\} = \delta_{ij}$ ...

$\{x_i, x_j\} = 0$

$\lambda_\alpha d\xi^\alpha \to p_i dx_i$ and $\theta_\alpha d\eta^a \to P_i dX_i$.

$\hookrightarrow \lambda_{x_i} = p_i$

So: $dF = p_i dx_i - P_i dX_i$ and...

Assumption: x_i & X_i are independent coords on phase space.

[if (x_i, p_i) uniquely determines point in phase space... so (x_i, X_i)].

Then: tempting to write $F_1(x_i, X_i, t)$, and demand:

$$p_i = \frac{\partial F_1}{\partial x_i} \quad \text{and} \quad P_i = -\frac{\partial F_1}{\partial X_i}$$

This is called a Type 1 CT, generated by F_1

Example 1: take phase space $\mathbb{R}^2 : (x,p)$.

$$F_1(x,X) = xX + \lambda(x^2 - X^2) \qquad [\lambda = \text{parameter}].$$

$$p = \frac{\partial F_1}{\partial x} = X + 2\lambda x \qquad \text{and} \qquad P = -\frac{\partial F_1}{\partial X} = -x + 2\lambda X$$

Re-solve: $X(x,p)$ and $P(x,p)$:

$$X = p - 2\lambda x \qquad \text{and} \qquad P = -x + 2\lambda(p - 2\lambda x) = -(1+4\lambda^2)x + 2\lambda p$$

$$\{X, P\} = \{p - 2\lambda x, 2\lambda p - (1+4\lambda^2)x\}$$
$$= -(1+4\lambda^2)\underbrace{\{p,x\}}_{-1} - 4\lambda^2 \underbrace{\{x,p\}}_{1} = 1. \quad :)$$

There are 2 natural choices of λ_α:

(A): $\lambda_{x_i} = p_i \quad \lambda_{p_i} = 0$ $\quad (p_i dx_i)$

(B) $\lambda_{p_i} = -x_i \quad \lambda_{x_i} = 0.$ $\quad (-x_i dp_i)$

$$\omega_{\alpha\beta} = \left(\begin{array}{c|c} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{array}\right)$$

Pick A vs. B for old + new coordinates $\rightarrow$ $2\times 2 = 4$ types of CT!

Type 1: $F_1(x, X, t)$ w/ $dF_1 = p_i dx_i - P_i dX_i$

Type 2: $F_2(x, P, t)$ w/ $dF_2 = p_i dx_i + X_i dP_i$

OK if (x, P) are ind. coords on phase space.

Type 3: $F_3(p, X, t)$ w/ $dF_3 = -x_i dp_i - P_i dX_i$

Type 4: $F_4(p, P, t)$ w/ $dF_4 = -x_i dp_i + X_i dP_i$

we'll actually use these.

Example 2: infinitesimal CTs are type 2!

e.g. identity $x_i \to X_i$ & $p_i \to P_i$

$$F_2(x_i, P_i) = x_i P_i + \varepsilon \cdot F(x, P)$$

$$p_i = \frac{\partial F_2}{\partial x_i} = P_i + \varepsilon \frac{\partial F}{\partial x_i} \qquad X_i = \frac{\partial F_2}{\partial P_i} = x_i + \varepsilon \frac{\partial F}{\partial P_i}$$

$$\hookrightarrow P_i \approx p_i - \varepsilon \frac{\partial F(x, \overset{\approx P}{p})}{\partial x_i} \qquad X_i = x_i + \varepsilon \frac{\partial F(x,p)}{\partial p_i}$$

$$P_i \approx p_i + \varepsilon \{p_i, F\} \qquad X_i = x_i + \varepsilon \{x_i, F\}$$

Time-dependent CTs:

Consider a Type 2 CT: $F_2(x_i, P_i, t)$

Claim: if $\underline{H' = H + \frac{\partial F_2}{\partial t}}$ ... all old formulas work!

Why? general G: $\frac{d G(x,p,t)}{dt} = \frac{\partial G}{\partial t} + \frac{\partial G}{\partial x_i}\dot{x}_i + \frac{\partial G}{\partial p_i}\dot{p}_i = \frac{\partial G}{\partial t} + \underset{\text{old coords.}}{\{G, H\}}$.

Now: plug in $G = X_i = \frac{\partial F_2}{\partial P_i}$:

$$\{X_i, H\} = \frac{\partial H}{\partial P_i}$$

$$\dot{X}_i = \frac{\partial}{\partial t}\frac{\partial F_2}{\partial P_i} + \underbrace{\{X_i, H\}}_{\text{CT's leave PB unchanged.}} \quad \text{If } H(X,P):$$

$$\hookrightarrow \dot{X}_i = \frac{\partial}{\partial P_i}\left[H + \frac{\partial F_2}{\partial t}\right] = \frac{\partial H'}{\partial P_i} \quad \text{if} \quad \underbrace{H' = H + \frac{\partial F_2}{\partial t}}_{\substack{\text{express } H'(X,P) \\ [\text{inverting } p_i(x,P)]}}.$$

Final note: for Type 1 CTs: $H' = H + \frac{\partial F_1}{\partial t}$.