PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 28
Hamilton-Jacobi equation

November 1

lec 27: Type 1 CT: $(x_i, p_i) \longrightarrow (X_i, P_i)$ where (x_i, X_i) form independent coords on phase space.

Generating function F_1: $dF_1 = p_i dx_i - P_i dX_i$ $\left(p_i = \frac{\partial F_1}{\partial x_i}\right)$

Hamiltonian for $X, P \ldots$: $H' = H + \frac{\partial F_1}{\partial t}$

Can we choose F_1 so that:

1. $H' = 0$.
2. (X_i, P_i) to be constants of motion.

1 $\Longrightarrow$ 2: $\dot{X}_i = \frac{\partial H'}{\partial P_i} = 0$ $\quad \dot{P}_i = -\frac{\partial H'}{\partial X_i} = 0.$

Each $(X_i, P_i) \longrightarrow$ one choice of init cond for Ham eq in x_i, p_i.

If so... find $x_i(X, P, t)$ and $p_i(X, P, t)$.

Such a special F_1 will be called S:

Then: $$0 = H' = \frac{\partial S}{\partial t} + H(x_i, \underbrace{p_i}_{= \frac{\partial S}{\partial x_i}}, t)$$

Hamilton-Jacobi: $$\underline{0 = \frac{\partial S}{\partial t} + H\left(x_i, \frac{\partial S}{\partial x_i}, t\right)} \longrightarrow n \text{ coords } x_i, t$$

If solved...

$S(x_i, X_i, t)$ depends on integration constants X_i

Problem: HJ is a partial differential equation

... much harder than ODE! !! ☹

Strategy: look for special solutions to HJ, enabled by "separation of variables"

Example 1: central force problem:

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

Legendre transform!

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r).$$

go to HJ

$$0 = \frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + V(r)$$

"Separation of variables":

$$S(r,\theta,t) = S_r(r) + \underbrace{S_\theta(\theta)}_{\downarrow} + S_t(t)$$

$$0 = \cdots + \frac{1}{2mr^2} S_\theta'(\theta)^2 \rightarrow$$ bad! both r, θ dep...

{Aside: S.O.V. for Laplace: $\phi(x,y) = X(x)\,Y(y)$:

$$\frac{1}{\phi}\nabla^2\phi = \frac{X''(x)}{X} + \frac{Y''(y)}{Y} = 0. \qquad \frac{X''}{X} = \lambda \qquad \frac{Y''}{Y} = -\lambda.$$

const.
}

Need $\frac{\partial S}{\partial t} = \text{const.}$ since no other t-dep.

So: $S_t = -Et$, AND $S_\theta = \ell\theta$

$\hookrightarrow$ Plug into HJ: $0 = -E + \frac{S_r'^2}{2m} + \frac{\ell^2}{2mr^2} + V(r)$

consistent since each term only depends on r.

$$S_r'^2 = 2mE - \frac{\ell^2}{r^2} - V(r)\cdot 2m$$

or $S_r(r) = \int dr \sqrt{2mE - \frac{\ell^2}{r^2} - 2mV(r)}$ Solve up to quadratures (HW 10)

"integration constants"

We've found: $S(r,\theta,t;\, E, \ell)$

Idea: treat E, ℓ as our X_i.

Now find other 2 const. of motion:

$$-P_E = \frac{\partial S}{\partial E} = -t + \frac{\partial S_r}{\partial E} = -t + \int \frac{dr \cdot 2m}{2\sqrt{2mE - \ell^2/r^2 - 2mV}}$$

$$-P_\ell = \frac{\partial S}{\partial \ell} = \theta + \frac{\partial S_r}{\partial \ell} = \theta - \int \frac{dr \cdot \ell/r^2}{\sqrt{2mE - 2mV - \ell^2/r^2}}$$

Example 2: particle in magnetic field

$$H = \frac{p_x^2}{2m} + \frac{(p_y + qBx)^2}{2m}$$

$$\hookrightarrow \quad 0 = \frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{1}{2m}\left(\frac{\partial S}{\partial y} + qBx\right)^2$$

Separate variables: $S = -Et + p_y y + W(x)$:

$$2mE = W'(x)^2 + (p_y + qBx)^2$$

so: $$W(x) = \int dx \sqrt{2mE - (p_y + qBx)^2}.$$

Find new conserved quantities:

$$\frac{\partial S}{\partial E} = -t + \int dx \frac{2m}{2\sqrt{2mE - (p_y+qBx)^2}}$$

let $p_y + qBx = \sqrt{2mE}\cos\theta$
$qB \cdot dx = -\sqrt{2mE}\sin\theta \, d\theta$

$$= -t - m\int \frac{\sqrt{2mE}\sin\theta \, d\theta}{qB\sqrt{2mE(1-\cos^2\theta)}}$$

$$\text{const.} = -t - \frac{m}{qB}\theta$$

$$\theta = -\frac{qB}{m}(t - t_0) \longrightarrow \cos\left(\frac{qB}{m}(t-t_0)\right) = \frac{p_y + qBx(t)}{\sqrt{2mE}}.$$

Then:

$$\text{const.} = \frac{\partial S}{\partial p_y} = y - \int dx \frac{p_y + qBx}{\sqrt{2mE - (p_y+qBx)^2}}$$

$$= y + \frac{\sqrt{2mE}}{qB}\sin\theta$$

consts

or... $$(x - x_0)^2 + (y - y_0)^2 = \frac{2mE}{(qB)^2}.$$

circular trajectories