PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 29
Action-angle variables

November 3

lec 27: Type 2 CT from $(x_i, p_i) \to (X_i, P_i)$
where (x_i, P_i) are independent.

$$dF_2 = p_i\, dx_i + X_i\, dP_i$$

$$p_i = \frac{\partial F_2}{\partial x_i} \qquad X_i = \frac{\partial F_2}{\partial P_i}$$

Goal today: time-independent type 2 CT
such that $H = H(P_i)$

Then: $\dot{X}_i = \underbrace{\frac{\partial H}{\partial P_i}}_{\text{const.} = \omega_i}$ and $\underbrace{\dot{P}_i = -\frac{\partial H}{\partial X_i} = 0}_{P_i = \text{constant.}}$

$X_i(t) = X_i(0) + \omega_i t$ $\qquad$ $P_i(t) = P_i(0)$.

integrable! (lec 31)

$\{P_i, P_j\} = 0$.

Note: we'll do this for $H(x_i, p_i)$ (t-ind.), and $\frac{\partial F_2}{\partial t} = 0$

If successful... exactly solved system.

These coords are special enough $\longrightarrow$ action-angle variables.

$$X_i \to \phi_A \quad \text{and} \quad P_i \to J_A$$

(use ABC to denote A-A variables)
NOT use summation convention on ABC...

How to find J_A?

Motivation: Suppose the H-J equation solved (by separation of vars)

$$S(x_i, t) \longmapsto \sum_A W_A(x_A) - Et \qquad \text{(Assume } H \text{ t-ind.)}$$

implicit integration const. $W_A(x_A; X's)$

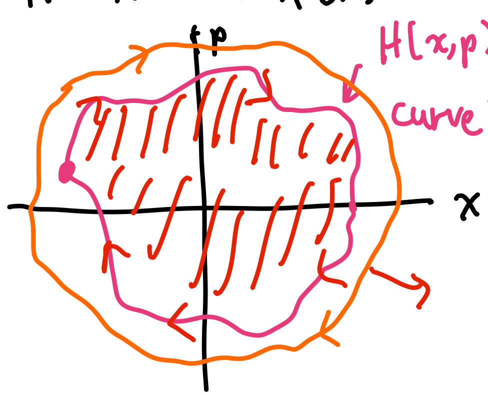
$$p_A = \frac{\partial W_A(x_A; X)}{\partial x_A}$$

const.

Idea: $J_A = \oint \frac{dx_A\, p_A}{2\pi}$ "action variables"

$$= \oint \frac{1}{2\pi} dx_A\, p_A = \oint \frac{1}{2\pi} dx_A \frac{\partial W_A(x_A, X)}{\partial x_A} = \oint \frac{1}{2\pi} dW_A$$

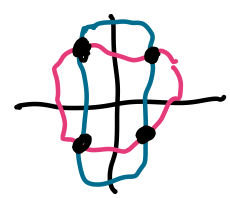
Picture / integration contour:
in two-dimensional phase space



$H(x,p) = E = \text{const.}$
curve = integration contour for J_A

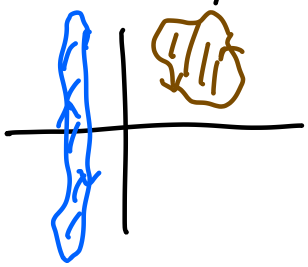
$J_A = \oint \frac{1}{2\pi} dW_A$ = Area enclosed by curve

Forbidden:



Try a curve w/ energy E': $J' > J$.

Possibility:

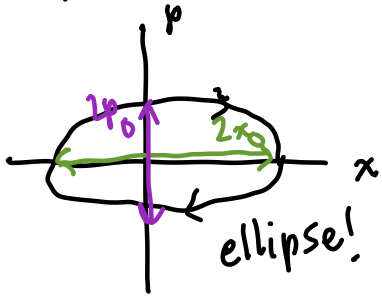


These could have equal J...

(see HW 11)

$\Rightarrow$ AA construction may not be global coords.

Example: harmonic oscillator $\qquad H = \dfrac{p^2}{2m} + \dfrac{1}{2} m\omega^2 x^2$



$$E = \frac{1}{2} m\omega^2 x_0^2 \Rightarrow x_0 = \sqrt{\frac{2E}{m\omega^2}}$$

$$E = \frac{p_0^2}{2m} \Rightarrow p_0 = \sqrt{2mE}$$

$$\text{Area} = \pi p_0 x_0 = \underset{\substack{\uparrow \\ \text{Area of unit circle}}}{\pi} \cdot \underbrace{p_0 \cdot x_0}$$

$\rightarrow$ rescale $x \to x/x_0$, $p \to p/p_0$.

$$J = \frac{1}{2\pi} \text{Area} = \frac{1}{2} p_0 x_0 = \frac{E}{\omega}$$

Thus $H(J) = \omega J$.

Hamilton's equations:

$$\dot{J} = -\frac{\partial H}{\partial \phi} = 0 \qquad \dot{\phi} = \frac{\partial H}{\partial J} = \omega$$

$J = \text{const.}$ $\qquad$ so $\phi(t) = \phi(0) + \omega t$.

$$\left.\begin{aligned} p &= p_0 \cos\phi \\ x &= x_0 \sin\phi \end{aligned}\right\} \rightarrow \begin{aligned} p &= \sqrt{2m\omega J}\cos\phi \\ x &= \sqrt{\frac{2J}{m\omega}}\sin\phi \end{aligned}$$

Check this is canonical by

$$\{x, p\} = \sqrt{\frac{2}{m\omega}}\sqrt{2m\omega}\,\{\sqrt{J}\sin\phi, \sqrt{J}\cos\phi\}$$

If $\{\phi, J\} = 1$, then:

$$\{x, p\} = 2\left[\frac{\partial(\sqrt{J}\sin\phi)}{\partial\phi}\frac{\partial(\sqrt{J}\cos\phi)}{\partial J} - \frac{\partial(\sqrt{J}\sin\phi)}{\partial J}\frac{\partial(\sqrt{J}\cos\phi)}{\partial\phi}\right] = 1.$$

Note: $\phi \sim \phi + 2\pi$ is periodic.

Recap: action (J_A) – angle (ϕ_A) variables have $H = H(J_A\text{'s})$

so $J_A =$ const. and $\phi_A(0) + \omega_A t$

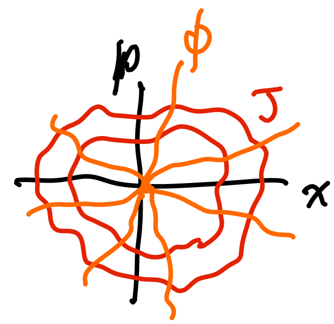
where $\omega_A = \frac{\partial H}{\partial J_A}$ (defined $\omega_A \ldots$)

If AA variables exist $\Longrightarrow$ system is solvable [integrable]

Why? AA > HJ ?

- nice interpretation $J_A = \oint \frac{1}{2\pi} dx_A \, p_A$
- (usually) $\phi_A \sim \phi_A + 2\pi$ periodic. . .

$\hookrightarrow$ geometric picture of phase space



Connection to QM: Bohr–Sommerfeld quantization.

If AA variables, then conjecture that

$$J_A = n_A \hbar \qquad n_A = 1, 2, 3, \ldots$$

Discretized energy: $E_{\{n_A\}} = H(n_1\hbar, \ldots, n_{max}\hbar)$

$\hookrightarrow$ $\approx$ WKB approximation.

but. . . $n_A \rightarrow n_A - 1/2$ (if $H(x,p)$ continuous)