PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 3
Noether's Theorem

September 1

Last time: continuous symmetry $\longrightarrow$ write L in invariant building blocks

Today: " " $\longrightarrow$ conserved quantity (Noether's Thm)

Example 1: translation. $x \to x + \varepsilon$

lec 2: $L = f(\dot{x}) = \boxed{\frac{1}{2} m \dot{x}^2} \rightarrow p = m\dot{x}$

Euler-Lagrange: $$\frac{\delta S}{\delta x} = 0 = \cancelto{0}{\frac{\partial L}{\partial x}} - \frac{d}{dt}\underbrace{\frac{\partial L}{\partial \dot{x}}}_{=p}$$

define: (momentum)

$\frac{dp}{dt} = 0$, i.e. p is conserved.

Generalize: continuous symmetry

$$t \to \tilde{t} = t + \varepsilon T(x_i, t) \qquad [\varepsilon \text{ infinitesimal}]$$

$$x_i(t) \to \tilde{x}_i(\tilde{t}) = x_i(t) + \varepsilon X_i(t, x_j)$$

$i = 1, \ldots, n$ $\to$ $$= x_i(\tilde{t}) + \varepsilon\left[X_i(\tilde{t}, x_j) - \dot{x}_i T(\tilde{t}, x)\right]$$

[E.g. translation: $X_1 = 1$, $T = 0$]

[rotation: $X_1 = x_2$ $X_2 = -x_1$

$x_1 \to x_1 + \underset{(\theta)}{\varepsilon} x_2$ $x_2 \to x_2 - \varepsilon x_1$]

Demand: $S[x_i(t)] = S[\tilde{x}_i(\tilde{t})] +$ boundary term

$\hookrightarrow L(x) = L(\tilde{x}) + \underbrace{\varepsilon \frac{d}{dt}\Phi}$

$$S[x_i] = S[\tilde{x}_i(\tilde{t})]$$

$$= \int_{\tilde{t}_i}^{\tilde{t}_f} d\tilde{t}\left[L(\tilde{x}_i, \dot{\tilde{x}}_i, \tilde{t}) + \varepsilon \frac{d\Phi}{d\tilde{t}}\right]$$

$$= \int_{t_i + \varepsilon T_i}^{t_f + \varepsilon T_f} d\tilde{t}\left[L\left(x_i + \varepsilon(X_i - \dot{x}_i T), \dot{x}_i + \varepsilon \frac{d}{dt}(X_i - \dot{x}_i T), \tilde{t}\right) + \varepsilon \frac{d\Phi}{dt}\right]$$

If S invariant: $\underline{\left.\frac{dS}{d\varepsilon}\right|_{\varepsilon=0} = 0}$

evaluate at t_f

$$\frac{dS}{d\varepsilon} = T_f L_f - T_i L_i + \underbrace{\int_{t_i}^{t_f} d\tilde{t}\, \frac{dL}{d\varepsilon}} + \Phi(t_f) - \Phi(t_i) = 0.$$

Goal: $K_f - K_i \to TL + K + \Phi$ conserved.

$$\frac{dL}{d\varepsilon} = \sum_{i=1}^{n}\left[\underbrace{\frac{\partial L}{\partial x_i}}_{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right)}(X_i - \dot{x}_i T) + \frac{\partial L}{\partial \dot{x}_i}\frac{d}{dt}(X_i - \dot{x}_i T)\right]$$

[repeated index summed]

$$= \frac{d}{dt}\left[\sum_{i=1}^{n}\frac{\partial L}{\partial \dot{x}_i}(X_i - \dot{x}_i T)\right] = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}_i}(X_i - \dot{x}_i T)\right]$$

Put together: cont. sym.

$$0 = \left[TL + \frac{\partial L}{\partial \dot{x}_i}(X_i - \dot{x}_i T) + \Phi\right]\Bigg|_{t_i}^{t_f}$$

$$\frac{d}{dt}\Big[\underbrace{TL + \frac{\partial L}{\partial \dot{x}_i}(X_i - \dot{x}_i T) + \Phi}_{Q}\Big] = 0 \quad \text{, on physical traj.}$$

Noether's Thm! continuous Symmetry $\Rightarrow$ conservation law

Above argument gives invariant BBs:

$$0 = \frac{d}{dt}(TL + \Phi) + \frac{dL}{d\varepsilon}\Big|_{\varepsilon=0}$$

$\downarrow$ no E-L, algebra

$$\boxed{0 = \frac{\partial L}{\partial x_i}X_i + \frac{\partial L}{\partial \dot{x}_i}(\dot{X}_i - \dot{T}\dot{x}_i) + \dot{T}L + \dot{\Phi}}$$

Example 1: translation symmetry:

$$\underbrace{x_1 \to x_1 + \varepsilon}_{X_1 = 1} \qquad \underbrace{t \to t}_{T=0} \qquad \underline{\Phi = 0}$$

$$Q = X_1 \frac{\partial L}{\partial \dot{x}_1} \to \frac{\partial L}{\partial \dot{x}_1} = \text{conserved} \quad (p_1)$$

e.g. $L = \frac{1}{2} m \dot{x}_i \dot{x}_i = \sum_{i=1}^{n} \frac{1}{2} m \dot{x}_i^2$; $\quad p_1 = m \dot{x}_1$

Example 2: rotational symmetry

$$\left.\begin{aligned} x_1 &\to x_1 + \underbrace{\varepsilon x_2}_{X_1} \\ x_2 &\to x_2 - \underbrace{\varepsilon x_1}_{X_2} \end{aligned}\right.$$

Conserved: $Q = \begin{matrix}\text{angular} \\ \text{momentum}\end{matrix} = \frac{\partial L}{\partial \dot{x}_1} x_2 - \frac{\partial L}{\partial \dot{x}_2} x_1$

Last time: $L = L(\underbrace{x_1^2 + x_2^2}_{I_1}, \underbrace{\dot{x}_1^2 + \dot{x}_2^2}_{I_2}, \underbrace{x\dot{y} - y\dot{x}}_{I_3})$

$$Q = \frac{\partial L}{\partial I_2}\left[x_2(2\dot{x}_1) - x_1(2\dot{x}_2)\right] + \frac{\partial L}{\partial I_3}\left[x_2(-x_2) - x_1 \cdot x_1\right]$$

$$= 2 I_3 \frac{\partial L}{\partial I_2} - I_1 \frac{\partial L}{\partial I_3}$$

Conceptually... solve for I_3 in terms of Q.

$$L(I_1, I_2, I_3) \to L(I_1, I_2, Q) \quad \leftarrow \text{not dynamical}$$

$$L(I_1, I_2)$$

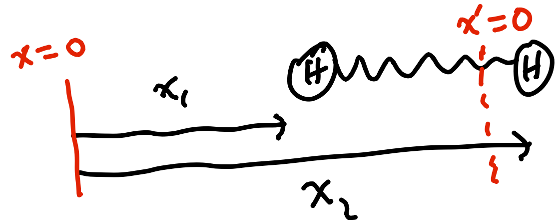
$L(x_1, x_2, \dot{x}_1, \dot{x}_2)$

$\hookrightarrow L(r, \dot{r}, \dot{\theta}) \to L(r, \dot{r}, Q)$

$\downarrow$

$L(r, \dot{r})$ one less DOF.

Example 3: interacting particles / H_2 molecule



<u>Translation symmetry:</u>

$$x_1 \to x_1 + \varepsilon$$
$$x_2 \to x_2 + \varepsilon$$

Step 1: invariant BBs:

$$\dot{x}_1, \ \dot{x}_2, \ x_1 - x_2$$

↳ Most general $L(\dot{x}_1, \dot{x}_2, x_1 - x_2)$

Taylor expand $\dot{x}$'s: $\frac{1}{2} m_g \dot{x}_1^2 + \frac{1}{2} m_g \dot{x}_2^2 + \cdots$

$- V(x_1 - x_2)$

Step 2: Noether:

$$Q = \frac{\partial L}{\partial \dot{x}_1} + \frac{\partial L}{\partial \dot{x}_2} = \text{const.}$$

→ Q = total mom. $= m_g \dot{x}_1 + m_g \dot{x}_2$

Step 3: change coords to remove "non-dynamical" center of mass:

$$x = x_1 - x_2$$

↳ $L(x, \dot{x}, Q)$

↳ $= \frac{1}{4} m \dot{x}^2 - V(x)$

relative molecular motion = 1d particle

(+ COM)