PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 30
Adiabatic theorem

November 6

lec 29: action-angle variables (ϕ, J) have $\underbrace{H = H(J)}_{\text{time-independent}}$

$\{\phi_A, J_B\} = \delta_{AB}$

if done: $\dot{\phi} = \frac{\partial H}{\partial J} = \overset{\text{const.}}{\omega}$ $\quad \dot{J} = -\frac{\partial H}{\partial \phi} = 0$, so $J = \text{const.}$

If found $\to$ integrable!

Today: Aside!

Adiabatic Theorem: if $H(J; t)$ changes very slowly in t, then $\dot{J} \simeq 0$

and so at each t, approx. using AA variables for H at t.

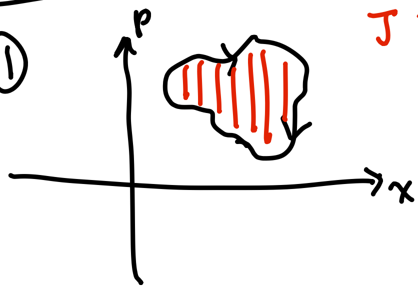
So: J is an "adiabatic invariant".

How slowly does H need to vary w/ t?

$\dot{\phi} = \omega \gg \frac{1}{\tau}$ where τ is time scale on which H varies

then adiabatic approx good.

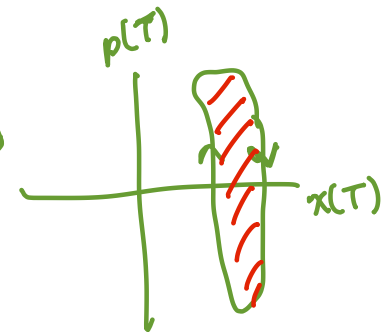
Proof sketch:

① 

$J = \frac{1}{2\pi} \text{Area} = \frac{1}{2\pi} \oint p\,dx$

↳ Stoke's Thm: $= \frac{1}{2\pi} \int dx\,dp \cdot \underbrace{\omega_{xp}}_{=1}$

Canonical transformation leaves PB & ω_{xp} invariant.

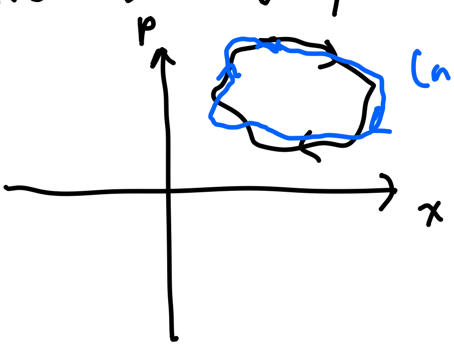


$\frac{1}{2\pi} \text{Area}_T = \frac{1}{2\pi} \int dx(T)\,dp(T) \cdot \underbrace{\omega_{x(T)p(T)}}_{=1}$

CTs don't change area (in 2d)

↳ Liouville's Thm.

② Now let H vary slowly ($\tau \gg \frac{2\pi}{\omega}$)



(next orbit) coming from $H(J) \to H(J, t = \frac{2\pi}{\omega})$

time evolution w/ $H(t)$ still generates CT...

by ①, blue/black curves enclose same Area $= 2\pi J$.

Adiabatic Thm: $J = \frac{1}{2\pi} \oint p\,dx \approx$ const.

"fixed t"

Example 1: Harmonic oscillator. $H = \frac{p^2}{2m} + \frac{1}{2} m \omega_0(t)^2 x^2$

- Vary $\omega_0(t)$ adiabatically (slowly). $\omega_0 \to 2\omega_0$.

How does amplitude of oscillations (x_0) change?

Find AA variables (lec 29): $J = E/\omega_0$

so if $\omega_0 \to 2\omega_0$, then $J \to J$, and $E \to 2E$.

Amplitude $E = \frac{1}{2} m\omega_0^2 x_0^2$

$$\downarrow \qquad \downarrow$$

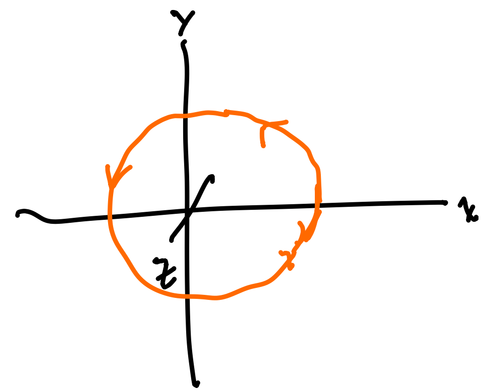
$2E = \frac{1}{2} m (4\omega_0^2)(x_0')^2$ and $x_0' = \frac{1}{\sqrt{2}} x_0$.

What if change ~~abrupt?~~ instantaneous

- If change happens when $p=0$, then $x_0 \to x_0$.
- If change when $x=0$, then $x_0 \to \frac{1}{2} x_0$ (E fixed)

<u>Example 2</u>: particles in B-fields

$$H = \frac{(p_i - qA_i)(p_i - qA_i)}{2m}$$



Suppose $\vec{B} = B_0 \hat{z}$. What's J?

$J = \frac{1}{2\pi} \oint [dx\, p_x + dy\, p_y]$ on orbit.

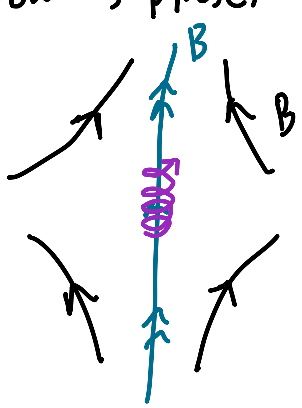
↳ use $J = L_z$ (ang. mom.) since in polar coords, $\dot{\theta} = \omega_c$

in: $$H = \frac{p_r^2}{2m} + \underbrace{\frac{(L_z - \frac{q}{2} B_0 r^2)^2}{2mr^2}} + \frac{p_z^2}{2m}$$

stable circular orbit if $V_{eff}(r)$ has minimum

when $L_z = \frac{qB_0}{2} r^2$ ⤻ $J = \frac{qB_0 r^2}{2} = \frac{1}{2\pi} q \Phi_B$ (orbit)

Now suppose:



particle drifts along field line,
"B_z" slowly changing.

Think of motion in z-direction as adiabatic...

Adiabatic Thm: circular orbit enclose Φ_B

or $B \cdot r^2 = \text{const}$.

Use energy conservation:

$$E = \underbrace{\tfrac{1}{2} m v_z^2}_{\text{along field line}} + \underbrace{\tfrac{1}{2} m v_\theta^2}_{\text{circular}}$$

$$v_\theta = \omega r = \frac{qB}{m} r$$

(Newton's Law)

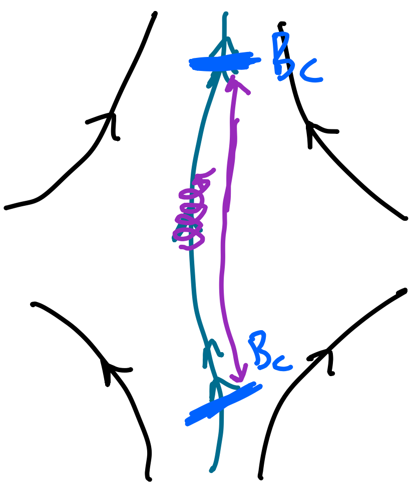
use adiabatic

const.

$$E = \frac{1}{2} m v_z^2 + \frac{qB}{2m} J$$

particle can't exceed

$$B_c = \frac{2mE}{qJ}$$



particle's trajectory bounces

→ "magnetic mirror"