PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 31
Integrable systems

November 8

Hamiltonian H is integrable if there exist n independent conserved I_A $(A=1,\dots,n)$ on $2n$-dim phase space. Obey:

(in involution) $\{I_A, I_B\} = 0$ for all A and B.

Always take $I_1 = H$ (if H is t-ind.)

Liouville's Integrability Thm: if surfaces of const. I_A are compact (don't go to ∞), then there are action-angle variables (ϕ_A, J_A) w/ $\{\phi_A, J_B\} = \delta_{AB}$ and $H = H(J)$.

A-A vars completely solve problem:

$$\dot{J}_A = -\frac{\partial H}{\partial \phi_A} = 0 \qquad \text{and} \qquad \dot{\phi}_A = \frac{\partial H}{\partial J_A} = \omega_A = \text{const.}$$

$$J_A = \text{const.} \qquad\qquad \phi_A(t) = \phi_A(0) + \omega_A t$$

Thus integrable system is:

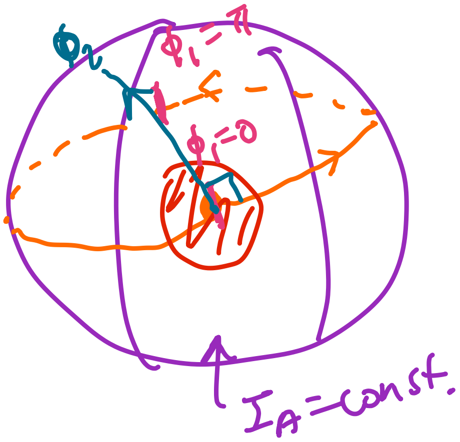
① exactly solved

② gives geometric picture for phase space.

Sketch LI Theorem's proof:

① By lec 24, each I_A generate CTs along surface of const. I_A.

(b/c $\delta I_A = \epsilon\{I_A, I_A\} = 0$)



Claim: there exists a $J_1(I_1, \dots, I_n)$ where flow returns to starting point. (in a local region)

Flow needs to return near start b/c surface is compact.

Then define angle variable ϕ_1 to be coord. along flow

Normalize ϕ_1 & J_1 so that $\phi_1 \sim \phi_1 + 2\pi$.

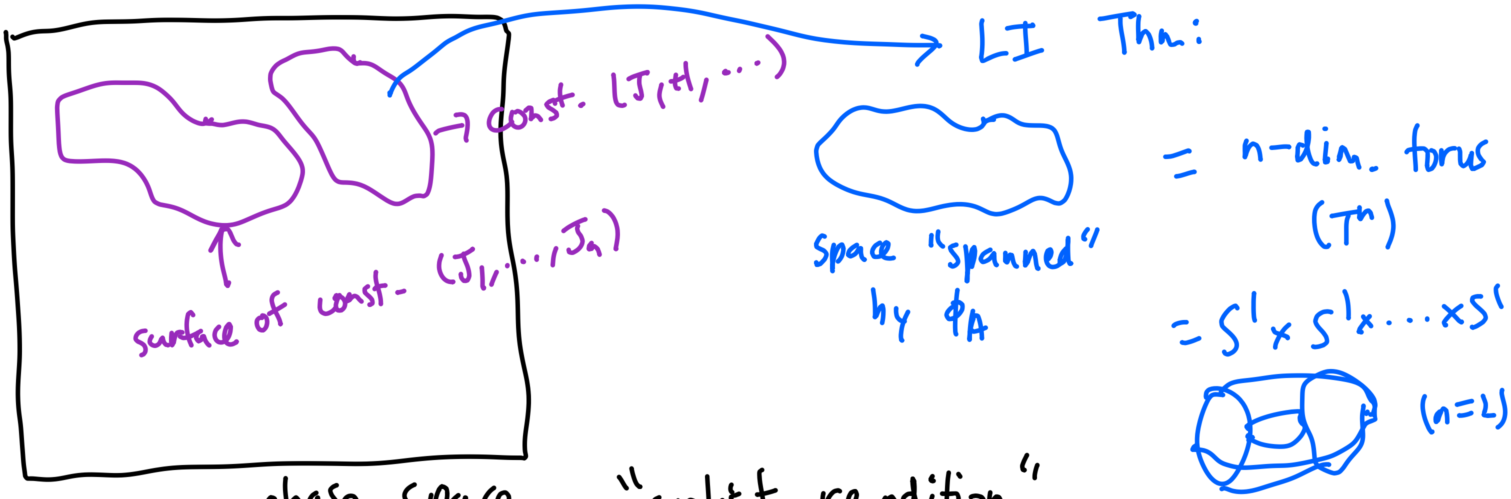
② Iterate... Choose $J_2(I_1, \dots, I_n)$ indep. of J_1 and choose to generate flow perp. to ϕ_1... give us ϕ_2

These flows preserve <u>all</u> I_A & J_A ... and

$\{\phi_1, J_2\} = 0$, $\{\phi_2, J_1\} = 0$.

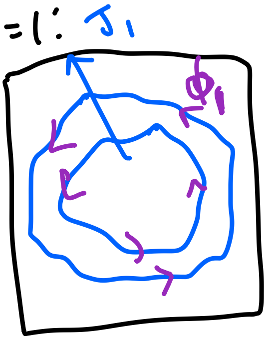
Normalize so $\{\phi_2, J_2\} = 1$, etc....

We can do this construction if I_A's are smooth...



phase space "artist rendition"

If n=1:



A-A as useful "polar" coords...

LI Thm: keeps working in 2n-dim phase space.

This is abstract... in practice:

- find AA vars by separating HJ eq?
- or stop w/ I_A's ... (see HW 11)

Well developed theory of integrability...

strongest in one spatial dim (lattice / field)

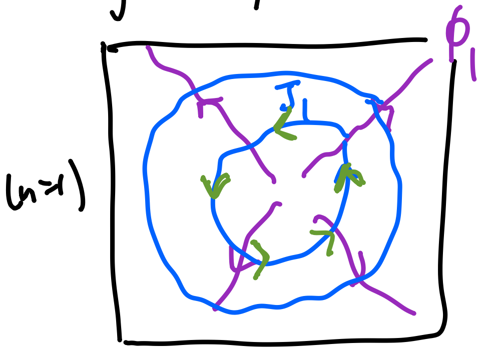
not much for $d>1$ except harmonic osc.

How can something NOT be integrable?

"Loophole" = I_A's might not be smooth (lec 34+)

Much of our knowledge of physics based on integrable systems, or perturbations of them.

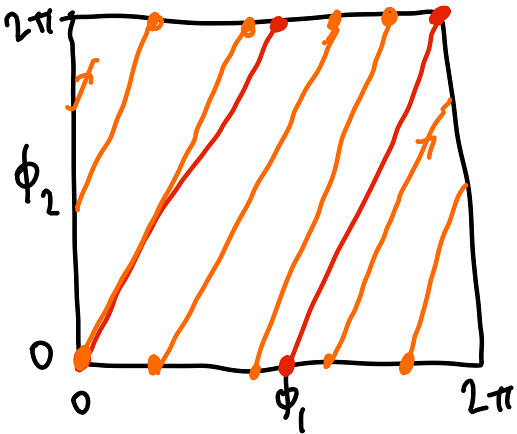
Integrability:



$$\dot{J}_A = 0$$

$$\underbrace{\dot{\phi}_A = \frac{\partial H}{\partial J_A} = \omega_A = \text{const.}}_{\text{wind around each circle of } T^n \text{ at frequency } \omega_A}$$

Now: $n=2$?



Draw T^2 as box w/ opposite edges ident.

① commensurate: $\frac{\omega_1}{\omega_2}$ = rational $(=\frac{1}{2})$
(ratio of integers)

② incommensurate: $\frac{\omega_1}{\omega_2}$ = irrational

Commensurate: trajectory = one-dimensional closed loop.

Incom: never return to origin, trajectory "fills up T^2".

for each point (ϕ_1, ϕ_2), we get w/in dist ε of point for any $\varepsilon > 0$. Trajectory is "$\underbrace{\text{dense}}$"

"ergodicity"

math fact:

$p + q \cdot \alpha$ can be as close as we want to any real # x.

↑ ↗ integers

↑ irrational (fix α!)

$$= \frac{1}{T}\int_0^T dt\, F(t) \to \text{avg of } F \text{ on } T^n.$$

$$= \frac{1}{(2\pi)^2}\int d\phi_1 d\phi_2\, F$$

Incom: seems like 1d trajectory → exploring 2 dim space?

Integrable system can traverse 1, 2, 3, ..., n-dim subspace of phase space.

Chaotic systems can explore 1.76...-dimensional subspace (Fractals!)