PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 32
Perturbation theory: one degree of freedom

November 10

Example: $H = \underbrace{\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2}_{H_0} + \underbrace{\varepsilon x^4}_{\varepsilon H_1}$ (take ε to be perturbatively small)

How small does ε need to be?

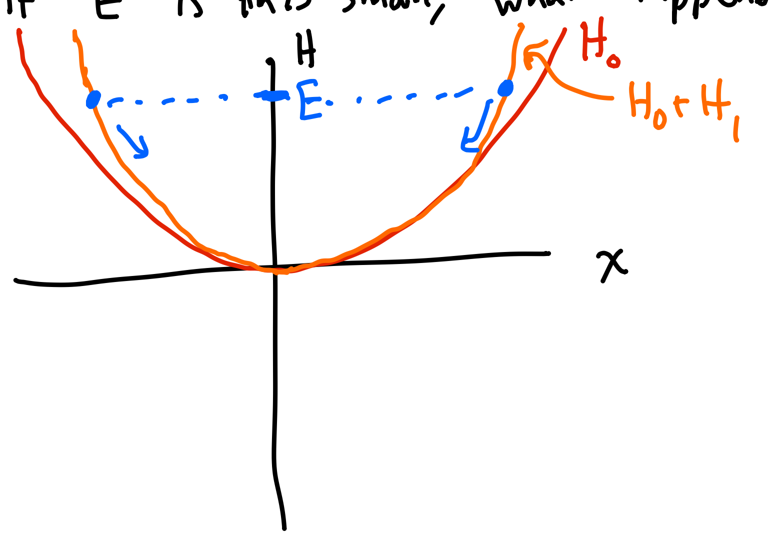
$$\varepsilon A^4 \ll \frac{1}{2}m\omega^2 A^2 \approx E.$$

$A^2 \ll \frac{m\omega^2}{2\varepsilon}$

A: amplitude of x

or $E \ll \frac{m^2\omega^4}{\varepsilon}$

If E is this small, what happens?



motion = periodic
oscillate btwn $\pm x$.

Goal today: H_0 is exactly solvable (integrable)
Build a perturbation theory that dynamics to "first order in ε."

Method 1: (literal interpretation?)

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots$$
$$p(t) = p_0(t) + \varepsilon p_1(t) + \varepsilon^2 p_2(t) + \cdots$$

Plug & chug: $\left[H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 + \varepsilon x^4\right]$

$$\dot{x} = \frac{\partial H}{\partial p} = p/m \qquad \dot{p} = -\frac{\partial H}{\partial x} = -m\omega^2 x - 4\varepsilon x^3$$

$$\dot{x}_0 + \varepsilon \dot{x}_1 + \cdots = \frac{p_0 + \varepsilon p_1 + \cdots}{m}$$

$$\dot{p}_0 + \varepsilon \dot{p}_1 + \cdots = -m\omega^2(x_0 + \varepsilon x_1) - 4\varepsilon(x_0 + \varepsilon x_1 + \cdots)^3$$

Collect orders in ε:

Order ε^0: $\dot{x}_0 = \frac{p_0}{m}$ and $\dot{p}_0 = -m\omega^2 x_0$

Solved by: $x_0 = A\cos(\omega t)$, $p_0 = -m\omega^2 A \sin(\omega t)$.

Order ε^1: $\dot{x}_1 = \frac{p_1}{m}$ and $\dot{p}_1 = -m\omega^2 x_1 - 4x_0^3$

$$\hookrightarrow x_1(t) = \frac{3A^3}{8\omega^2}\left[\underbrace{4\omega t \cos(\omega t)} + \sin(3\omega t) - 7\sin(\omega t)\right].$$

NOT good: diverges as $t \to \infty$, PT fails!

Why? $x(t) = A\cos\left[(\omega_0 + \varepsilon\omega_1 + \cdots)t\right] + \cdots$

$$\simeq A\cos(\omega_0 t) - \underbrace{A\sin(\omega t)\cdot \varepsilon\omega_1 t}_{\text{similar divergence!}} + \cdots$$

Method 2: AA variables. Better behaved?
Find new AA vars that keep problem integrable.

General theory: AA vars (ϕ_0, J_0) for H_0.

$$H = H_0(J_0) + \varepsilon H_1(J_0, \phi_0)$$

Goal: find Type 2 CT to new AA vars (J, ϕ) such that

$$H(\phi_0, J_0) \to H(J)$$

Look for perturbatively small CT:

$$J = J_0 + \varepsilon J_1 + \cdots$$

$$\phi = \phi_0 + \varepsilon \phi_1 + \cdots$$

ensures $J = J_0$, $\phi = \phi_0$ at $\varepsilon = 0$.

focus

Generating function for CT: $S(\phi_0, J) = \boxed{\phi_0 J} + \boxed{\varepsilon S_1} + \varepsilon^2 S_2 + \cdots$

Then: $$J_0 = \frac{\partial S}{\partial \phi_0} = J + \varepsilon \frac{\partial S_1}{\partial \phi_0} + \cdots$$

$$\phi = \frac{\partial S}{\partial J} = \phi_0 + \varepsilon \frac{\partial S_1}{\partial J} + \cdots$$

Hamiltonian: $H(J) = H_0(J_0) + \varepsilon H_1(J_0, \phi_0)$

$$H(J) = H_0\left(J + \varepsilon \frac{\partial S_1}{\partial \phi_0} + \cdots\right) + \varepsilon H_1\left(J + \varepsilon \frac{\partial S_1}{\partial \phi_0}, \phi_0\right)$$

At order ε^0: $H(J) = H_0(J)$ [do nothing]

At order ε: $$H(J) = H_0(J) + \varepsilon \frac{\partial S_1}{\partial \phi_0} \underbrace{\frac{\partial H_0(J)}{\partial J}}_{\omega_0} + \varepsilon H_1(J, \phi_0)$$

$$H(J) = H_0(J) + \varepsilon \underbrace{\left[\omega_0 \frac{\partial S_1}{\partial \phi_0} + H_1\right]}$$

only a function of J?

Brute force solution: use periodicity of ϕ_0 to expand H_1 & S_1 as Fourier series:

$$S_1(J,\phi_0) = \sum_{m=-\infty}^{\infty} e^{im\phi_0} s_m(J) \quad , \quad H_1 = \sum_{m=-\infty}^{\infty} e^{im\phi_0} h_m(J)$$

$$\downarrow$$

$$\frac{\partial S_1}{\partial \phi_0} = \sum_{m\neq 0} e^{im\phi_0} \cdot i m s_m(J)$$

Goal: $\omega_0 \frac{\partial S_1}{\partial \phi_0} + H_1$ only have $m=0$ harmonic.

$$\hookrightarrow \sum_{m=-\infty}^{\infty} \left[i m s_m(J)\omega_0 + h_m(J) \right] e^{im\phi_0}$$

Pick $s_m = 0$ $(m=0)$

$s_m = -\dfrac{h_m}{im\omega_0}$ so terms above cancel if $m \neq 0$.

Thus, found new AA variables!

$$H(J) = H_0(J) + \varepsilon h_0(J)$$

first order correction is ϕ_0-ind. part of pert.

$$\hookrightarrow h_0(J_0) = \frac{1}{2\pi}\int_0^{2\pi} d\phi_0 \, H_1(J_0, \phi_0)$$

$= t$- avg of H_1

Calculate perturbed frequency!

$$\omega = \frac{\partial H}{\partial J} = \omega_0 + \varepsilon \frac{\partial h_0}{\partial J} + \cdots$$

Example again: $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \varepsilon x^4$

Convert to AA variables of H_0 (lec 29).

$$H = \omega J_0 + \varepsilon\left(\sqrt{\frac{2J_0}{m\omega}}\cos\phi_0\right)^4$$

Calculate

$$h_0(J) = \frac{1}{2\pi}\int_0^{2\pi} d\phi_0 \, H_1(J, \phi_0) = \frac{1}{2\pi}\int_0^{2\pi} d\phi_0 \, \frac{4J^2}{m^2\omega^2}\cdot\cos^4\phi_0$$

$$= \frac{3}{2}\frac{J^2}{m^2\omega^2}$$

New Hamiltonian: $H(J) = \omega J + \varepsilon\cdot\frac{3}{2}\frac{J^2}{m^2\omega^2}$

New oscillation frequency:

$$\omega_{new} = \frac{\partial H}{\partial J} = \omega + 3\varepsilon\frac{J}{m^2\omega^2} \rightarrow \omega + \underbrace{3\varepsilon\frac{E}{m^2\omega^3}}$$

small if $\omega \gg \frac{E\varepsilon}{m^2\omega^3}$ ✓